# Homework Assignment 1 <br> Due via canvas Feb 9th 

## SDS 384-11 Theoretical Statistics <br> Please do not add your name to the HW submission. Also do not add collaborators here or in the comments section of Canvas.

1. (2 pt) Given densities $f_{n}$ and $g_{n}$ with respect to some measure $\mu$, let $X$ be distributed according to the distribution with density $f_{n}$. Define the likelihood ratio $L_{n}(X)$ as $L_{n}(X)=g_{n}(X) / f_{n}(X)$ for $f_{n}(X)>0$, and $L_{n}(X)=1$, if $f_{n}(X)=g_{n}(X)=$ 0 and $L_{n}(X)=\infty$ otherwise. Show that the likelihood ratio is a uniformly tight sequence. First note that $E L_{n}(X)=\int_{f_{n}(x)>0} g_{n}(x) / f_{n}(x) d x \leq 1$, and hence by Markov's inequality, for any $\epsilon, P\left(L_{n}(X)>1 / \epsilon\right) \leq \epsilon$. This establishes tightness.
2. $(1+2+3)$ We will do some examples of convergence in distribution and convergence in probability here.
(a) Let $X_{n} \sim N(0, n)$. Prove that $X_{n}=O_{p}(\sqrt{n})$ and $o_{P}(n)$. Since $X_{n} / \sqrt{n} \xrightarrow{d}$ $N(0,1)$, we see that $X_{n} / \sqrt{n}=O_{P}(1)$ and hence $X_{n}=O_{P}(\sqrt{n})$. As for the last part, $P\left(X_{n} / n \geq t\right) \leq 1 / n t^{2}$ and hence $X_{n} / n=o_{P}(1)$.
(b) Let $\left\{X_{n}\right\}$ be independent r.v's such that $P\left(X_{n}=n^{\alpha}\right)=1 / n$ and $P\left(X_{n}=0\right)=$ $1-1 / n$ for $n \geq 1$, where $\alpha \in(-\infty, \infty)$ is a constant. For what values of $\alpha$, will you have $X_{n} \xrightarrow{q . m} 0$ ? For what values will you have $X_{n} \xrightarrow{p} 0$ ?
Convergence in quadratic mean:

$$
\mathrm{E}\left[\left|X_{n}\right|^{2}\right]=\frac{n^{2 \alpha}}{n}
$$

The above will converge to zero if $2 \alpha<1$, or $\alpha<\frac{1}{2}$.
Convergence in Probability:
For $\epsilon \geq n^{\alpha}$ we have $\operatorname{Pr}\left(\left|X_{n}\right|>\epsilon\right)=0$. For $\epsilon<n^{\alpha}$ we have $\operatorname{Pr}\left(\left|X_{n}\right|>\epsilon\right)=\frac{1}{n}$. This probability converges to zero for all values of $\alpha$.
(c) Consider the average of $n$ i.i.d random variables $X_{1}, \ldots, X_{n}$ with $E\left[X_{1}\right]=\mu$ and $E\left[\left|X_{1}\right|\right]<\infty$. Write true or false.
i. $\bar{X}_{n}=o_{P}(1)$

We know that $\bar{X}_{n}$ converges to $\mu$ in probability. If $\mu \neq 0, \bar{X}_{n}=o_{P}(1)$ is false.
ii. $\exp \left(\bar{X}_{n}-\mu\right)=o_{P}(1)$

Solution. We know that $\bar{X}_{n}-\mu$ converges to 0 in probability. By continuous mapping, If $\exp \left(\bar{X}_{n}-\mu\right) \xrightarrow{P} 1$. So false.
iii. $\left(\bar{X}_{n}-\mu\right)^{2}=O_{P}(1 / n)$

Fix $\epsilon>0$. Now $P(\left(\bar{X}_{n}-\mu\right)^{2} \geq \underbrace{\frac{\sigma^{2}}{\epsilon}}_{M_{\epsilon}}) \leq \epsilon$. So, its true.
3. $(2+4+1)$ Consider random variables $X_{1}, \ldots, X_{n}$ be IID r.v's with mean $\mu$ and variance $\sigma^{2}:=\operatorname{var}\left(X_{i}\right)$. We will use the following statistic to estimate $\theta=\mu^{2}$.

$$
\hat{\theta}=\frac{1}{\binom{n}{2}} \sum_{i<j} X_{i} X_{j}
$$

(a) Find constants $C_{1}, C_{2}$ where

$$
\hat{\theta}-\mu^{2}=\frac{C_{1}}{\binom{n}{2}} \sum_{i<j}\left(X_{i}-\mu\right)\left(X_{j}-\mu\right)+\frac{C_{2} \mu}{n} \sum_{i}\left(X_{i}-\mu\right)
$$

We have,

$$
\begin{aligned}
\frac{1}{\binom{n}{2}} \sum_{i<j} X_{i} X_{j}-\mu^{2} & =\frac{1}{n(n-1)} \sum_{i \neq j} X_{i} X_{j}-\mu^{2} \\
& =\frac{1}{n(n-1)} \sum_{i \neq j}\left(\left(X_{i}-\mu\right)\left(X_{j}-\mu\right)+\mu\left(X_{i}-\mu\right)+\mu\left(X_{j}-\mu\right)\right) \\
& =\underbrace{\frac{1}{n(n-1)} \sum_{i \neq j}\left(X_{i}-\mu\right)\left(X_{j}-\mu\right)}_{T_{1}}+\underbrace{\frac{2}{n} \mu \sum_{i}\left(X_{i}-\mu\right)}_{T_{2}}
\end{aligned}
$$

Thus, $C_{1}=1, C_{2}=2$.
(b) Show that the first term is $O_{P}(1 / n)$ and the second term is $O_{P}(1 / \sqrt{n})$.

Observe that,

$$
\operatorname{var}\left(T_{1}\right)=\frac{1}{n^{2}(n-1)^{2}}\left(\sum_{i \neq j, k \neq \ell} E\left(X_{i}-\mu\right)\left(X_{j}-\mu\right)\left(X_{k}-\mu\right)\left(X_{\ell}-\mu\right)\right)
$$

But in the above sum, all tuples with $i \neq j \neq k \neq \ell$ are zero. All tuples with $i \neq j=k \neq \ell$ are also zero. The only nonzero terms arise from $i=k \neq j=\ell$ or $i=\ell \neq j=k$. And there are $O\left(n^{2}\right)$ such terms all with expectation $\sigma^{4}$. Thus the variance of $T_{1}$ is $O\left(1 / n^{2}\right)$. We also see that

$$
\operatorname{var}\left(T_{2}\right)=O(1 / n)
$$

Now note that for any sequence of mean zero random variables $X_{n}, Y_{n}=$ $X_{n} / \sqrt{\operatorname{var}\left(X_{n}\right)}=O_{P}(1)$. This is because,

$$
\sup _{n} P\left(\left|Y_{n}\right| \geq 1 / \sqrt{\epsilon}\right) \leq \epsilon
$$

Therefore, $T_{1}=O_{P}(1 / n)$ and $T_{2}=O_{P}(1 / \sqrt{n})$.
(c) Argue that $\hat{\theta} \xrightarrow{P} \mu^{2}$.

Since $\hat{\theta}-\mu^{2}=o_{P}(1)$, this is proved.
4. $(3+2+2+3)$ If $X_{n} \xrightarrow{d} X \sim \operatorname{Poisson}(\lambda)$, is it necessarily true that $E\left[g\left(X_{n}\right)\right] \rightarrow$ $E[g(X)]$ ? Prove your answer when you believe the answer is true. When you believe it is "not necessarily true", provide a counter-example.
(a) $g(x)=1(x \in(0,10))$

This is not necessarily true since $g(x)$ is not continuous at $x=0$. Consider the sequence of random variables

$$
X_{n}=X+\frac{1}{n}
$$

Clearly, $X_{n} \xrightarrow{p} X$ (and consequently $X_{n} \xrightarrow{d} X$ ). However, since $X \sim \mathcal{P}(\lambda)$, therefore $X \geq 0$. Therefore, $\forall n \geq 1, X_{n}>0$. Therefore,

$$
g\left(X_{n}\right)=\left\{\begin{array}{l}
1 \text { for } X<10-\frac{1}{n} \\
0 \text { otherwise }
\end{array}\right.
$$

and

$$
g(X)=\left\{\begin{array}{l}
1 \text { for } X<10 \text { and } X \geq 1 \\
0 \text { otherwise }
\end{array}\right.
$$

Therefore,

$$
E g\left(X_{n}\right)=P\left(X<10-\frac{1}{n}\right) \rightarrow P(X<10)
$$

but

$$
E g(X)=P(X<10)-P(X=0)=P(X<10)-e^{-\lambda}
$$

(b) $g(x)=e^{-x^{2}}$

True by Portmanteau thm.
(c) $g(x)=\operatorname{sgn}(\cos (x))[\operatorname{sgn}(x)=1$ if $x>0,-1$ if $x<0$ and 0 if $x=0$.]

Also true by Portmanteau thm, since $g(x)$ is bounded and the discontintuity points are all at odd multiples of $\pi / 2$, which are not intergers, and hence the limiting random variable has zero probability mass on this set.
(d) $g(x)=x$

Not necessarily true since $g(x)$ is not bounded. Consider a counter example:

$$
X_{n}= \begin{cases}X & \text { with probability } 1-1 / n \\ n & \text { with probability } 1 / n\end{cases}
$$

But $E X_{n}=E X(1-1 / n)+1 \rightarrow E X+1$.
5. (1+4) Consider $X_{n}$ Uniform on $\{1 / n, 2 / n, \ldots, 1\}$. Let $X \sim \operatorname{Uniform}([0,1])$. For the questions below, either give a proof or a counter-example.
(a) Does $X_{n} \xrightarrow{d} X$ ?

Yes. If $t \leq 1, P\left(X_{n} \leq t\right)=\frac{\lfloor\min (t n, n)\rfloor}{n} \rightarrow t$.
(b) Does $X_{n} \xrightarrow{P} X$ ?

No, first, we need to define $X_{n}$ and $X$ on the same probability space to even start thinking about convergence in probability. But we will show with a counter example that even with such a construction we can couple $X_{n}$ and $X$ such that $X_{n} \xrightarrow{d} X$ but $X_{n}$ does not converge in probability to $X$.
First define $Y_{n}=\lceil n X\rceil / n$. Note that $Y_{n}$ is a discrete Uniform. Now define $X_{n}=1+1 / n-Y_{n}$. Clearly, this is also a discrete uniform, and hence converges in distribution to $X$, but what about convergence in probability?

$$
\begin{aligned}
& P\left(X_{n}-X \geq 1 / 2\right)=P\left(1+1 / n-Y_{n}-X \geq 1 / 2\right) \\
& =P\left(Y_{n}+X \leq 1 / 2+1 / n\right)=P(\lceil n X\rceil+n X \leq n / 2+1) \geq P(X \leq 1 / 4)
\end{aligned}
$$

which does not converge to zero.

