Homework Assignment 1

Due via canvas Feb 11th

SDS 384-11 Theoretical Statistics

Please **do not** add your name to the HW submission.

1. Given densities f_n and g_n with respect to some measure μ, let X be distributed according to the distribution with density f_n. Define the likelihood ratio $L_n(X)$ as $L_n(X) = g_n(X)/f_n(X)$ for $f_n(X) > 0$, and $L_n(X) = 1$, if $f_n(X) = g_n(X) = 0$ and $L_n(X) = \infty$ otherwise. Show that the likelihood ratio is a uniformly tight sequence.

2. We will do some examples of convergence in distribution and convergence in probability here.

   (a) Let $X_n \sim N(0, n)$. Prove that $X_n = O_p(\sqrt{n})$ and $o_P(n)$.

   (b) Let $\{X_n\}$ be independent r.v's such that $P(X_n = n^\alpha) = 1/n$ and $P(X_n = 0) = 1 - 1/n$ for $n \geq 1$, where $\alpha \in (-\infty, \infty)$ is a constant. For what values of α, will you have $X_n \stackrel{q.m}{\to} 0$? For what values will you have $X_n \stackrel{p}{\to} 0$?

   (c) Consider the average of n i.i.d random variables $X_1, \ldots, X_n$ with $E[X_1] = \mu$ and $E[|X_1|] < \infty$. Write true or false.

      i. $\bar{X}_n = o_P(1)$

      ii. $\exp(\bar{X}_n - \mu) = o_P(1)$

      iii. $(\bar{X}_n - \mu)^2 = O_P(1/n)$

3. Consider random variables $X_1, \ldots, X_n$ be IID r.v's with mean μ and variance $\sigma^2 := \text{var}(X_i)$. We will use the following statistic to estimate $\theta = \mu^2$.

$$\hat{\theta} = \frac{1}{\binom{n}{2}} \sum_{i<j} X_i X_j$$

   (a) Find constants C_1, C_2 where

$$\hat{\theta} - \mu^2 = \frac{C_1}{\binom{n}{2}} \sum_{i<j} (X_i - \mu)(X_j - \mu) + \frac{C_2 \mu}{n} \sum_i (X_i - \mu)$$

   (b) Show that the first term is $O_P(1/n)$ and the second term is $O_P(1/\sqrt{n})$.

   (c) Argue that $\hat{\theta} \stackrel{P}{\to} \mu^2$.

4. If $X_n \stackrel{d}{\to} X \sim Poisson(\lambda)$, is it necessarily true that $E[g(X_n)] \to E[g(X)]$? Prove your answer when you believe the answer is true. When you believe it is "not necessarily true", provide a counter-example.

   (a) $g(x) = 1(x \in (0, 10))$

(b) $g(x) = e^{-x^2}$

(c) $g(x) = sgn(cos(x))$ [$sgn(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.]

(d) $g(x) = x$

5. Consider X_n Uniform on $\{1/n, 2/n, \ldots, 1\}$. Let $X \sim$ Uniform($[0,1]$). For the questions below, either give a proof or a counter-example.

   (a) Does $X_n \xrightarrow{d} X$?

   (b) Does $X_n \xrightarrow{P} X$?