Homework Assignment 1
Due via canvas Feb 17th

SDS 384-11 Theoretical Statistics

1. Consider a sequence of iid random variables \( \{X_n\} \) such that \( X_i \sim Beta(\theta, 1) \), where \( \theta > 0 \). Let \( \bar{X}_n \) denote the sample mean. The method of moments estimator of \( \theta \) is \( \hat{\theta}_n = \frac{\bar{X}_n}{1 - \bar{X}_n} \). Derive the asymptotic distribution of \( \sqrt{n}(\hat{\theta}_n - \theta) \).

2. We will do some examples of convergence in distribution and convergence in probability here.
   (a) Let \( X_n \sim N(0, 1/n) \). Does \( X_n \xrightarrow{d} 0? \)
   (b) Let \( \{X_n\} \) be independent r.v’s such that \( P(X_n = n^\alpha) = 1/n \) and \( P(X_n = 0) = 1 - 1/n \) for \( n \geq 1 \), where \( \alpha \in (-\infty, \infty) \) is a constant. For what values of \( \alpha \), will you have \( X_n \xrightarrow{q.m.} 0? \) For what values will you have \( X_n \xrightarrow{p} 0? \)

3. If \( X_n \xrightarrow{d} X \sim Poisson(\lambda) \), is it necessarily true that \( E[g(X_n)] \xrightarrow{} E[g(X)]? \)
   (a) \( g(x) = 1(x \in (0, 10)) \)
   (b) \( g(x) = e^{-x^2} \)
   (c) \( g(x) = \text{sgn}(\cos(x)) \) \([\text{sgn}(x) = 1 \text{ if } x > 0, -1 \text{ if } x < 0 \text{ and } 0 \text{ if } x = 0.\]
   (d) \( g(x) = x \)

4. Let \( X_1, \ldots, X_n \) be independent r.v’s with mean zero and variance \( \sigma_i^2 := E[X_i^2] \) and \( s_n^2 = \sum_i \sigma_i^2 \). If \( \exists \delta > 0 \) s.t. as \( n \to \infty \),
   \[
   \frac{\sum_i E|X_i|^{2+\delta}}{s_n^{2+\delta}} \to 0,
   \]
   then \( \sum_i X_i/s_n \) converges weakly to the standard normal.

5. Recall the converse of the Lindeberg Feller theorem. We will gather some intuition about that here. Let \( X_1, \ldots, X_n \) be independent r.v’s with mean zero and variance \( \sigma_i^2 := E[X_i^2] \) and \( s_n^2 = \sum_i \sigma_i^2 \).
   (a) If \( \max_i \sigma_i^2/s_n^2 \) does not converge to zero as \( n \to \infty \), then the Lindeberg condition does not hold.
   (b) Construct an example where the above is true, but still we have \( \sum_i X_i/s_n \) converges weakly to \( N(0, 1) \). This shows that the Lindeberg condition is not necessary. You can show this by showing that the moment generating function converges to that of a standard normal.