THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 384 11: Theoretical Statistics

Lecture 12: Uniform Law of Large Numbers- VC dimension

Purnamrita Sarkar
Department of Statistics and Data Science
The University of Texas at Austin

Rademacher Complexity for general function classes

Recall that for $|f(x)| \leq 1$,

$$
\begin{aligned}
\|\hat{P}_n - P\|_{\mathcal{F}} &\leq 2\mathcal{R}_{\mathcal{F}} + \epsilon = 2E[E[\sup_{f\in\mathcal{F}} \sum_i \epsilon_i f(X_i)/n]|X] + \epsilon \\
&\leq 2E\sqrt{\frac{2\log(|\mathcal{F}(X_1^n) \cup -\mathcal{F}(X)|)}{n}} + \epsilon \\
&\leq \sqrt{\frac{8\log 2\max_X |\mathcal{F}(X_1^n)|}{n}} + \epsilon
\end{aligned}
$$

- How do I control $|\mathcal{F}(X_1^n)|$?
- How big is $\max_X |\mathcal{F}(X_1^n)|$?
- Let us focus on binary functions, i.e. $f(X_i) \in \{0, 1\}$

Growth function

Definition

For a binary valued function class $\mathcal{F}$, the growth function is:

$$\Pi_{\mathcal{F}}(n) = \max\{|\mathcal{F}(x_1^n)| x_1, \ldots, x_n \in \mathcal{X}\}$$

- $\mathcal{X}$ could be $\mathbb{R}^d$.
- $\mathcal{R}_{\mathcal{F}} \leq \sqrt{\frac{2\log(2\Pi_{\mathcal{F}}(n))}{n}}$
- $\Pi_{\mathcal{F}}(n) \leq 2^n$ (which is not really useful)
- We are looking for $\Pi_{\mathcal{F}}(n)$ growing polynomially with n.
    - Because then $\|\hat{P}_n - P\|_{\mathcal{F}} \xrightarrow{P} 0$

Vapnik-Chervonenkis Dimension

Definition

A dichotomy of a set S is a partition of S into two disjoint subsets.

Definition (In words)

A set of instances S is shattered by a binary function class $\mathcal{F}$ iff for every dichotomy of S, there is some function in $\mathcal{F}$ consistent with this dichotomy.

Definition (In math)

A binary function class $\mathcal{F}$ shatters $(x_1, \ldots, x_d) \subseteq \mathcal{X}$, implies that $|\mathcal{F}(x_1^d)| = 2^d$.

Vapnik-Chervonenkis Dimension

Definition

The VC dimension of a binary function class $\mathcal{F}$ is given by

$$\begin{aligned} d_{VC}(F) &= \max\{d : \text{some } x_1, \ldots, x_d \in \mathcal{X} \text{ is shattered by } \mathcal{F}\} \\ &= \max\{d : \Pi_{\mathcal{F}}(d) = 2^d\} \end{aligned}$$

- If the VC dimension of a function class is small, then $\Pi_{\mathcal{F}}(n)$ is small.

Sauer's lemma

Theorem

If $d_{VC}(F) \leq d$, then

$$\Pi_F(n) \leq \sum_{i=0}^{d} \binom{n}{i}.$$

If $n \geq d$, the latter sum is no more than $(en/d)^d$.

- So we have the growth function is either polynomially growing with d, or 2^n.

$$\Pi_F(n) = \begin{cases} = 2^n & \text{If } n \leq d \\ \leq \left(\frac{en}{d}\right)^d & \text{If } n > d \end{cases}$$

VC dimension-examples

Example

Let $\mathcal{F} = \{1_{(-\infty,t]} : t \in \mathbb{R}\}$ and $\mathcal{X} = \mathbb{R}$. Then $d_{VC}(\mathcal{F}) = 1$.

- First show that there exists some configuration of one point, which can be shattered by $\mathcal{F}$.
  - For any point x, if x has label 1, use $t > x$
  - If x has label 0, use $t < x$.
- Now show that there exists no two points which can be shattered by $\mathcal{F}$. (this takes a bit of an argument in more complex cases.)
  - For any two points (x, y) the labeling $(0, 1)$ cannot be achieved by any function in $\mathcal{F}$.

VC dimension-examples

Example

Let $\mathcal{F}$ be linear classifiers in $\mathcal{X} = \mathbb{R}^2$. Then $d_{VC}(\mathcal{F}) = 3$.

- First show that there exists some configuration of 3 points, which can be shattered by $\mathcal{F}$.
  - Purna draws picture, and if you miss class, you can easily draw a picture to see this.
- Now show that there exists no 4 points which can be shattered by $\mathcal{F}$. (this takes a bit of an argument.)

VC dimension-examples

Example

Let $\mathcal{F}$ be linear classifiers in $\mathcal{X} = \mathbb{R}^2$. Then $d_{VC}(\mathcal{F}) = 3$.

- Now show that there exists no 4 points which can be shattered by $\mathcal{F}$. (this takes a bit of an argument.)
    - Take 4 non-collinear points. If they are collinear, it is easy to find label configurations which cannot be shattered by a linear classifier.
    - The convex hull of these points will either be a triangle, or a quadrilateral.
    - In case the convex hull is a triangle, and there is a third point inside the convex hull, give all the points on the hull label 1 and the one inside label 0.
    - If three points are collinear or the convex hull is a quadrilateral, then just label the consecutive points with alternative labels.

VC dimension: decision stumps in 2D

Example

Let $\mathcal{F}$ be decision stumps in two dimensions. Then $d_{VC}(\mathcal{F}) = 3$.

- Show that there exists three points in 2D which can be shattered by this function class. Purna draws picture.
- Now show that no four points in 2D can be shattered.

VC dimension: decision stumps in 2D

Example

Let $\mathcal{F}$ be decision stumps in two dimensions. Then $d_{VC}(\mathcal{F}) = 3$.

- Case 1: all 4 points are collinear. Easy to see that this cannot be shattered, since $1, 0, 1, 0$ is not achievable.
- Case 2: the convex hull of the 4 points is a triangle.
  - Case 2a: the 4th point is on a side of this triangle. So three points are collinear, and a $1, 0, 1$ labeling cannot be achieved by a decision stump.
  - Case 2b: the 4th point is inside. Label all the points outside as 1 and the 4th as 0. This cannot be achieved.
- Case 3: the convex hull is a quadrilateral. Just label $1, 0, 1, 0$ along the hull and this cannot be achieved.

VC dimension: rectangles

Example

Let $\mathcal{F}$ be classifiers which classify the interior (plus boundary) as one of axis aligned rectangles in $\mathcal{X} = \mathbb{R}^2$. Then $d_{VC}(\mathcal{F}) = 4$.

- This is on your homework.

Sauer's lemma proof - using shifting

- For a fixed $x_1, \ldots, x_n$, consider the following table.
- Let $\mathcal{F} = \{f_1, \ldots, f_5\}$ and let $\mathcal{F}$ have VC dimension d.

	x_1	x_2	x_3	x_4	x_5
f_1	0	1	0	1	1
f_2	1	0	0	1	1
f_3	1	1	1	0	1
f_4	0	1	1	0	0
f_5	0	0	0	1	0

- $|\mathcal{F}|$ is the number of distinct rows of the above table.

Sauer's lemma proof [Courtesy: P. Frankl]

- Consider the following shifting operation of the table.
- You start shifting columns from left to right.
- For each column, change a 1 to a zero unless it leads to a row which is already in the table.

	x_1	x_2	x_3	x_4	x_5
f_1	0	1	0	1	1
f_2	1	0	0	1	1
f_3	1	1	1	0	1
f_4	0	1	1	0	0
f_5	0	0	0	1	0

	x_1	x_2	x_3	x_4	x_5
f_1	0	1	0	0	0
f_2	0	0	0	0	1
f_3	0	0	1	0	1
f_4	0	0	1	0	0
f_5	0	0	0	0	0

Sauer's lemma proof [Courtesy: P. Frankl]

- This operation is done column after column until nothing can be shifted.
- The number of unique rows does not change.
- An all zero column implies that any subset containing that datapoint is not shattered.
- Consider a row with some 1's. Let S be the set of points with the 1's.
    - Every configuration with any of these 1's turned into zeros is a row in this table.
    - In other words S is shattered by $\mathcal{F}$.

Sauer's lemma proof [Courtesy: P. Frankl]

- The column shifting never shatters a set that was not shattered already, i.e. a set of points can go from shattered to un-shattered but not the other way around.
  - If a column is all zeros after shifting, then any subset containing that datapoint is not shattered.
  - Say you have gone through column i. The table (or function class) was F before you started shifting column i and is F' after.
  - Say subset A ($i \in A$) is shattered in F'. We will show that it was also shattered in F
    - Each row with 1 in column i of F' is also there in F
    - Consider a row with 0 in column i in F'.

    $\rightarrow$ Since A is shattered by F', the same pattern (constrained to A) with a 1 in column i must also be in F'.

    $\rightarrow$ And therefore, the same pattern (constrained to A) must be there in F with a 0 in column I (since you could not shift it down.)
  - So A must be shattered by F as well.

Sauer's lemma proof [Courtesy: P. Frankl]

- So shifting cannot increase VC dimension.
- Each row has at most d ones.
- The final step is how many rows can the shifted table (and hence the original table) have?
- Well the upper bound is the same as number of length n binary strings you can make with at most d ones.