THE UNIVERSITY OF TEXAS AT AUSTIN
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SDS 384 11: Theoretical Statistics

Lecture 15: Uniform Law of Large Numbers-Applications

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Application-Random matrix singular value

Theorem

Consider a random matrix $M = (\xi_{ij})_{i,j \in [n]}$ *where* ξ_{ij} *are standard normal random variables.*

$$P(\|M\|_{op} \geq A\sqrt{n}) \leq C \exp(-cAn)$$

where c, C *are absolute constants and* $A \geq C$.

- This works for symmetric wigner ensembles and hermitian matrices as well.

Operator norm

- Let $S_n := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$
- $\|M\|_{op} := \sup_{x \in S_n} \|Mx\|$
- First note that we have

$$P(\|Mx\| \geq A\sqrt{n}) \leq C \exp(-cAn)$$

- This is because for each row M_i^T, we have

$$M_i^T x \sim Subgaussian(1), (M_i^T x)^2 - 1 \sim Subexponential(2, 4)$$

- $\|Mx\|^2 - n \sim Subexponential(2\sqrt{n}, 4)$

Recall sub-exponential random variables?

Theorem

Let X be a sub-exponential random variable with parameters (ν, b). Then,

$$P(X \geq \mu + t) \leq \begin{cases} e^{-\frac{t^2}{2\nu^2}} & \text{if } 0 \leq t \leq \frac{\nu^2}{b} \\ e^{-\frac{t}{2b}} & \text{if } t \geq \frac{\nu^2}{b} \end{cases}$$

- $P(\|Mx\|^2 - n \geq Cn) \leq e^{-Cn/8}$, $C > 1$.

Can I just use an Union bound?

- Not really.
- But I can form a $1/2$ cover of S_n.
- Find $\mathcal{C} = \{x^1, \ldots, x^N\}$ such that for all $x \in S_n$, $\exists x^i \in S$ $\|x - x^i\| \leq 1/2$.
- Consider $y \in S$ such that $\|My\| = \|M\|_{op}$. Let x^i be a member of the $1/2$ cover s.t. $\|y - x^i\| \leq 1/2$
- So $\|M(y - x^i)\| \leq \|M\|_{op}/2$ and $\|M(y - x^i)\| \geq \|My\| - \|Mx^i\| \geq \|M\|_{op} - \|Mx^i\|$.
- Hence $\|Mx^i\| \geq \|M\|_{op}/2$

Using the covering number

$$\begin{aligned} P(\|M\|_{op} \geq \sqrt{(C+1)n}) &\leq P(\exists x^i \in \mathcal{C}, \|Mx^i\| \geq \sqrt{(C+1)n}/2) \\ &\leq |\mathcal{C}| P(\|Mx^i\| \geq \sqrt{(C+1)n}/4) \\ &\leq |\mathcal{C}| P(\|Mx^i\|^2 - n \geq (C-3)n/4) \\ C > 7 \text{ gives } (C-3)n/4 \geq \nu^2/b \qquad &\leq |\mathcal{C}| \exp(-(C-3)n/32) \end{aligned}$$

- ϵ covering number of the unit ball in n dimensions is bounded by $(1+2/\epsilon)^n$

$$\begin{aligned} P(\|M\|_{op} \geq \sqrt{(C+1)n}) &\leq 5^n \exp(-(C-3)n/32) \\ &\leq \exp(-n((C-3)/32 - 1.6)) \end{aligned}$$

- So C will have to be something like 55!!