THE UNIVERSITY OF TEXAS AT AUSTIN
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SDS 384 11: Theoretical Statistics

Lecture 16: Uniform Law of Large Numbers- Dudley's chaining Introduction

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Rademacher complexity of function classes

Example

Suppose $\mathcal{F}$ is a class parametric functions $\mathcal{F} := \{f(\theta, .) : \theta \in B_2\}$, where B_2 is the unit L_2 ball in $\mathbb{R}^d$. Assume that $\mathcal{F}$ is closed under negation. f is L Lipschitz w.r.t. the Euclidean distance on Θ, i.e. $|f(\theta, .) - f(\theta', .)| \leq L\|\theta - \theta'\|_2$.

$$\mathcal{R}_n(\mathcal{F}) = O\left(L\sqrt{\frac{d\log(Ln)}{n}}\right)$$

- How do we do this?
- Using covering numbers. But we need to define a bunch of stuff first.

A Stochastic Process

- Consider a set $\mathcal{T} \subseteq \mathcal{R}^{\boldsymbol{d}}$.
- The family of random variables $\{X_\theta : \theta \in \mathcal{T}\}$ define a Stochastic process indexed by $\mathcal{T}$.
- We are often interested in the behavior of this process given its dependence on the structure of the set $\mathcal{T}$.
- In the other direction, we want to know the structure of $\mathcal{T}$ given the behavior of this process.

Gaussian and Rademacher processes

Definition

A canonical Gaussian process is indexed by $\mathcal{T}$ is defined as:

$$G_\theta := \langle z, \theta \rangle = \sum_k z_k \theta_k,$$

where $z_k \overset{\text{iid}}{\sim} N(0,1)$. The supremum $\mathcal{G}(\mathcal{T}) := E_z[\sup_{\theta \in \mathcal{T}} G_\theta]$ is the Gaussian complexity of $\mathcal{T}$.

Rademacher complexity

- Replacing the iid standard normal variables by iid Rademacher random variables gives a Rademacher process $\{R_\theta, \theta \in \mathcal{T}\}$, where

$$R_\theta := \langle \epsilon, \theta \rangle = \sum_k \epsilon_k \theta_k, \qquad \text{where } \epsilon_k \overset{\text{iid}}{\sim} \mathit{Uniform}\{-1, 1\}$$

- $\mathcal{R}(\mathcal{T}) := E_\epsilon[\sup_{\theta \in \mathcal{T}} R_\theta]$ is called the Rademacher complexity of $\mathcal{T}$.

How does this relate to the former notions of Rademacher complexity?

- Recall that
  $\mathcal{R}_{\mathcal{F}} := E[\sup_{f \in \mathcal{F}} |\sum_i \epsilon_i f(X_i)|] = E[E[\sup_{f \in \mathcal{F}} |\sum_i \epsilon_i f(X_i)||X_1, \ldots, X_n]]$
- Now the inner expectation can be upper bounded by
  $E_\epsilon \sup_{\theta \in \mathcal{T} \bigcup -\mathcal{T}} \sum_i \epsilon_i \theta_i$, where $\mathcal{T} \subseteq \mathbb{R}^n$ can be written as

$$\mathcal{T} = \{(f(X_1), \ldots, f(X_n)) | f \in \mathcal{F}\}$$

Relationship

Theorem

For $\mathcal{T} \in \mathbb{R}^d$,

$$\mathcal{R}(\mathcal{T}) \leq \sqrt{\frac{\pi}{2}}\mathcal{G}(\mathcal{T}) \leq c\sqrt{\log d}\mathcal{R}(\mathcal{T})$$

- This is showing that there can be there are some sets where the Gaussian complexity can be substantially larger than the Rademacher complexity.
- We will in fact give an example.

Proof (of first inequality)

$$
\begin{aligned}
\mathcal{G}(\mathcal{T}) &= E \sup_{\theta \in \mathcal{T}} \sum_i z_i \theta_i \\
&= E \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i |z_i| \theta_i \\
&= E_\epsilon E_z \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i |z_i| \theta_i \\
&\geq E_\epsilon \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i E|z_i| \theta_i \\
&= \sqrt{\frac{2}{\pi}} \mathcal{R}(\mathcal{T})
\end{aligned}
$$

Proof (of second inequality)

Theorem (Ledoux-Talagrand contraction (simple form))

Consider n 1-Lipschitz functions ϕ_i.

$$E \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i \phi_i(\theta_i) \leq E \sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i \theta_i$$

Proof (of second inequality)

Proof.

$$
\begin{aligned}
\mathcal{G}(\mathcal{T}) &= E \sup_{\theta \in \mathcal{T}} \sum_i z_i \theta_i = E \sup_{\theta \in \mathcal{T}} \|z\|_\infty \sum_i \frac{z_i}{\|z\|_\infty} \theta_i \\
&= E_z E_\epsilon \left[\sup_{\theta \in \mathcal{T}} \|z\|_\infty \sum_i \epsilon_i \frac{|z_i|}{\|z\|_\infty} \theta_i | z_1, \ldots, z_n \right] \\
&= E_z \left(\|z\|_\infty E_\epsilon \left[\sup_{\theta \in \mathcal{T}} \sum_i \epsilon_i \frac{|z_i|}{\|z\|_\infty} \theta_i | z_1, \ldots, z_n \right] \right) \\
&\leq E_z \sup_{\theta \in \mathcal{T}} \|z\|_\infty E_\epsilon \sum_i \epsilon_i \theta_i
\end{aligned}
$$

Last step follows from the contraction argument. □

Proof of Ledoux-Talagrand contraction

Proof.

$$E \sup_{\theta \in \mathcal{T}} \underbrace{\sum_i \epsilon_i \phi_i(\theta_i)}_{h_n(\theta)} = E_\epsilon \sup_\theta \left(\sum_{i=1}^{n-1} \epsilon_i \phi_i(\theta_i) + \epsilon_n \phi_n(\theta_n) \right)$$

$$= E_{\epsilon_1^{n-1}} E_{\epsilon_n} \sup_\theta (h_{n-1}(\theta) + \epsilon_n \phi_n(\theta_n))$$

$$= E_{\epsilon_1^{n-1}} \left(\frac{1}{2} \sup_\theta (h_{n-1}(\theta) + \phi_n(\theta_n)) + \frac{1}{2} \sup_\theta (h_{n-1}(\theta) - \phi_n(\theta_n)) \right)$$

$$= E_{\epsilon_1^{n-1}} \left(\frac{1}{2} (h_{n-1}(\theta^*) + \phi_n(\theta_n^*)) + \frac{1}{2} (h_{n-1}(\tilde{\theta}) - \phi_n(\tilde{\theta}_n)) \right)$$

$$= E_{\epsilon_1^{n-1}} \left(\frac{1}{2} (h_{n-1}(\theta^*) + h_{n-1}(\tilde{\theta})) + \phi_n(\theta_n^*) - \phi_n(\tilde{\theta}_n) \right)$$

$$\leq E_{\epsilon_1^{n-1}} \left(\frac{1}{2} (h_{n-1}(\theta^*) + h_{n-1}(\tilde{\theta})) + s(\theta_n^* - \tilde{\theta}_n) \right)$$

Proof of Ledoux-Talagrand contraction

Proof.

$$E \sup_{\theta \in \mathcal{T}} \underbrace{\sum_i \epsilon_i \phi_i(\theta_i)}_{h_n(\theta)} = E_\epsilon \sup_\theta \left(\sum_{i=1}^{n-1} \epsilon_i \phi_i(\theta_i) + \epsilon_n \phi_n(\theta_n) \right)$$

$$\le E_{\epsilon_1^{n-1}} \left(\frac{1}{2}(h_{n-1}(\theta^*) + h_{n-1}(\tilde{\theta})) + s(\theta_n^* - \tilde{\theta}_n) \right)$$

$$= E_{\epsilon_1^{n-1}} \left(\frac{1}{2}(h_{n-1}(\theta^*) + s\theta_n^*) + \frac{1}{2}(h_{n-1}(\tilde{\theta}) - s\tilde{\theta}_n) \right)$$

$$\le E_{\epsilon_1^{n-1}} \left(\frac{1}{2} \sup_\theta (h_{n-1}(\theta) + s\theta_n) + \frac{1}{2} \sup_\theta (h_{n-1}(\theta) - s\theta_n) \right)$$

$$\le E_{\epsilon_1^{n-1}} E_{\epsilon_n} \left(\sup_\theta (h_{n-1}(\theta) + \epsilon_n \theta_n) \right)$$

□

Example

Example

Consider the L_1 ball in $\mathcal{R}^d$ denoted by B_1^d.

$$\mathcal{R}(B_1^d) = 1, \mathcal{G}(B_1^d) \leq \sqrt{2 \log d}$$

- $\mathcal{R}(B_1^d) = E[\sup_{\|\theta\|_1 \leq 1} \sum_i \theta_i \epsilon_i] = E[\|\epsilon\|_\infty] = 1$
- Similarly, $\mathcal{G}(B_1^d) = E[\|z\|_\infty]$

Recall the finite class lemma?

Theorem

Consider z with independent standard normal components.

$$E \max_{a \in A} < z, a > \leq \max_{a \in A} \|a\| \sqrt{2 \log |A|}$$

- In our case, $A = \{e_i, i \in [d]\}$, $e_i(j) = \pm 1(j = i)$, $|A| = 2d$ and $\max_{a \in A} \|a\| = 1$.
- This gives a weaker bound on the Gaussian complexity.

A sub-gaussian process

Definition

A stochastic process $\theta \to X_\theta$ with indexing set T is sub-Gaussian w.r.t a metric d_X if $\forall \theta, \theta' \in T$ and $\lambda \in \mathbb{R}$,

$$E \exp(\lambda(X_\theta - X'_\theta)) \leq \exp\left(\frac{\lambda^2 d_X(\theta, \theta')^2}{2}\right)$$

- This immediately implies the following tail bound.

$$P(|X_\theta - X_{\theta'}| \geq t) \leq 2 \exp\left(-\frac{t^2}{2 d_X(\theta, \theta')^2}\right)$$

Upper bound by 1 step discretization

Theorem

(1-step discretization bound). Let $\{X_\theta, \theta \in \mathcal{T}\}$ be a zero-mean sub-Gaussian process with respect to the metric d_X. Then for any $\delta > 0$, we have

$$E\left[\sup_{\theta,\theta'\in\mathcal{T}}(X_\theta - X_{\theta'})\right] \leq 2E\left[\sup_{\substack{\theta,\theta'\in\mathcal{T}\\ d_X(\theta,\theta')\leq\delta}}(X_\theta - X_{\theta'})\right] + 2D\sqrt{\log N(\delta;\mathcal{T},d_X)},$$

where $D := \max_{\theta,\theta'\in\Theta} d_X(\theta,\theta')$.

- The mean zero condition gives us:
  $E[\sup_{\theta\in\mathcal{T}} X_\theta] = E[\sup_{\theta\in\mathcal{T}}(X_\theta - X_{\theta_0})] \leq E[\sup_{\theta,\theta'\in\mathcal{T}}(X_\theta - X_{\theta'})]$

Tradeoff

$$E\left[\sup_{\theta,\theta'\in\mathcal{T}}(X_\theta - X_{\theta'})\right] \leq 2\,\underbrace{E\left[\sup_{\substack{\theta,\theta'\in\mathcal{T}\\ d_X(\theta,\theta')\leq\delta}}(X_\theta - X_{\theta'})\right]}_{\text{Approximation error}} + 4\underbrace{\sqrt{D^2\log N(\delta;\mathcal{T},d_X)}}_{\text{Estimation error}}$$

- As $\delta \to 0$, the cover becomes more refined, and so the approximation error decays to zero.
- But the estimation error grows.
- In practice the δ can be chosen to achieve the optimal trade-off between two terms.

Proof

- Choose a δ cover T.
- For $\theta, \theta' \in \mathcal{T}$, let $\theta^1, \theta^2 \in T$ such that $d_X(\theta, \theta^1) \leq \delta$ and $d_X(\theta', \theta^2) \leq \delta$.

$$X_\theta - X_{\theta'} = (X_\theta - X_{\theta^1}) + (X_{\theta^1} - X_{\theta^2}) + (X_{\theta^2} - X_{\theta'})$$
$$\leq 2 \sup_{\substack{\theta, \theta' \in \mathcal{T} \\ d_X(\theta, \theta') \leq \delta}} (X_\theta - X_{\theta'}) + \sup_{\theta^i, \theta^j \in T} (X_{\theta^i} - X_{\theta^j})$$

- But note that $X_{\theta^1} - X_{\theta^2} \sim \mathit{Subgaussian}(d_X(\theta^1, \theta^2))$..

Finite class lemma for subgaussian processes

Theorem

Consider X_θ sub-gaussian w.r.t d on $\mathcal{T}$ and A is a set of pairs from $\mathcal{T}$.

$$E \max_{(\theta,\theta')\in A} (X_\theta - X_{\theta'}) \leq D\sqrt{2\log|A|},$$

where $D := \max_{(\theta,\theta')\in A} d_X(\theta,\theta')$.

Finite class lemma

$$\begin{aligned}\exp\left(\lambda E \max_{(\theta,\theta')\in A}(X_\theta - X_{\theta'})\right) &\leq E\exp\left(\lambda \max_{(\theta,\theta')\in A}(X_\theta - X_{\theta'})\right) \\ &= \max_{(\theta,\theta')\in A} E\exp(\lambda(X_\theta - X_{\theta'})) \\ &\leq \sum_{(\theta,\theta')\in A} \exp\left(\frac{\lambda^2 d_X(\theta,\theta')^2}{2}\right) \\ &\leq |A|\exp\left(\frac{\lambda^2 D^2}{2}\right)\end{aligned}$$

- Now optimize over λ.

Finishing the proof

$$X_\theta - X_{\theta'} \leq 2 \sup_{\substack{\theta,\theta' \in \mathcal{T} \\ d_X(\theta,\theta') \leq \delta}} (X_\theta - X_{\theta'}) + \sup_{\theta^i,\theta^j \in T} (X_{\theta^1} - X_{\theta^2})$$

$$E\left[\sup_{\theta,\theta' \in \mathcal{T}} (X_\theta - X_{\theta'})\right] \leq 2E\left[\sup_{\substack{\theta,\theta' \in \mathcal{T} \\ d_X(\theta,\theta') \leq \delta}} (X_\theta - X_{\theta'})\right] + E\left[\sup_{\theta^i,\theta^j \in T} (X_{\theta^1} - X_{\theta^2})\right]$$

$$\leq 2E\left[\sup_{\substack{\theta,\theta' \in \mathcal{T} \\ d_X(\theta,\theta') \leq \delta}} (X_\theta - X_{\theta'})\right] + D\sqrt{2 \log N(\delta; \mathcal{T}, d_X)^2}$$

Revisiting: smoothly parametrized class

Example

Suppose $\mathcal{F}$ is a class parametric functions $\mathcal{F} := \{f(\theta, .) : \theta \in B_2\}$, where B_2 is the unit L_2 ball in $\mathbb{R}^d$. Assume that $\mathcal{F}$ is closed under negation. f is L Lipschitz w.r.t. the Euclidean distance on Θ, i.e. $|f(\theta, .) - f(\theta', .)| \leq L\|\theta - \theta'\|_2$.

$$\mathcal{R}_n(\mathcal{F}) = O\left(L\sqrt{\frac{d \log(Ln)}{n}}\right)$$

Proof

- Denote $f(\theta, X_1^n)$ as the vector $(f(\theta, X_1), \ldots, f(\theta, X_n))$.
- Recall that $n\mathcal{R}_n(\mathcal{F}) = E\left[\sup_{f \in \mathcal{F}} \langle \epsilon, f(\theta, X_1^n) \rangle\right] = E\left[\sup_{\theta \in \Theta} \langle \epsilon, f(\theta, X_1^n) \rangle\right]$
- The process $f(\theta, X_1^n) \to \langle \epsilon, f(\theta, X_1^n) \rangle =: Y_\theta$ is mean zero subgaussian.
- Note that $Y_\theta - Y_\theta' \sim \mathit{Subgaussian}(d_X(\theta, \theta'))$
- We have:

$$d_X(\theta, \theta')^2 = \|f(\theta, X_1^n) - f(\theta', X_1^n)\|^2 \leq nL^2 \|\theta - \theta'\|_2^2$$

  - So it is $L\sqrt{n}$ Lipschitz.

Proof

- Also,

$$n\mathcal{R}_n(\mathcal{F}) = E[\sup_{\theta \in \Theta} (Y_\theta - Y_{\theta'})] \leq E[\sup_{\theta, \theta' \in \Theta} (Y_\theta - Y_{\theta'})]$$

-

$$n\mathcal{R}_n(\mathcal{F}) \leq 2\,E \underbrace{\sup_{\substack{d_X(\theta,\theta') \leq \delta \\ \theta,\theta' \in \Theta}} (Y_\theta - Y'_\theta)}_{A} + 2D\sqrt{\log N(\delta; \mathcal{F}(\Theta, X_1^n), d_X)}$$

- $A \leq \delta E \left[\sup_{\|v\|_2 = 1} \langle \epsilon, v \rangle \right] \leq \delta \sqrt{n}$
- $D = \sup_{\theta, \theta'} d_X(\theta, \theta') = 2L\sqrt{n}$

Proof

- $N(\delta; \mathcal{F}, d_X) \leq N(\delta/L\sqrt{n}, \Theta, \|.\|_2) \leq \left(1 + \frac{L\sqrt{n}}{\delta}\right)^d$
- Finally,
$$\mathcal{R}_n(\mathcal{F}) \leq \frac{4\delta}{\sqrt{n}} + 4L\sqrt{\frac{d\log(1 + L\sqrt{n}/\delta)}{n}}$$
- Setting $\delta = 1$ gives:
$$\mathcal{R}_n(\mathcal{F}) \leq \frac{4L}{\sqrt{n}} + 4L\sqrt{\frac{d\log(1 + L\sqrt{n})}{n}}$$

Examples:Nonparametric functions

Example

Suppose $\mathcal{F}$ is a class of L Lipschitz functions which are supported on $[0,1]$ and $f(0)=0$. Note that $\mathcal{F}$ is closed under negation. f is L Lipschitz i.e. $|f(x)-f(x')| \leq L|x-x'| \; \forall x, x' \in [0,1]$.

$$\mathcal{R}_n(\mathcal{F}) = O\left(\frac{L}{n}\right)^{1/3}$$

Examples:Nonparametric functions

- Consider the process $f(X_1^n) \to \langle \epsilon, f(X_1^n) \rangle = Y_f$
- $Y_f - Y_{f'} \sim subGaussian(\|f(X_1^n) - f'(X_1^n)\|_2)$
- So $d_Y(f, f') = \|f(X_1^n) - f'(X_1^n)\|_2 \leq \sqrt{n}\|f - f'\|_\infty$
- The diameter is $D = \sup_{f,f' \in \mathcal{F}(X_1^n)} d_X(f, f') \leq 2L\sqrt{n}$
- So, $N(\delta, \mathcal{F}(X_1^n), \|.\|_2) \leq N(\delta/\sqrt{n}, \mathcal{F}, \|.\|_\infty)$

$$
\begin{aligned}
n\mathcal{R}_n(\mathcal{F}) &\leq E[\sup_{f \in \mathcal{F}(X_1^n)} Y_f] \leq E[\sup_{f,f' \in \mathcal{F}(X_1^n)} (Y_f - Y_{f'})] \\
&\leq 2E\left[\sup_{d_Y(f,f') \leq \delta} (Y_f - Y_{f'})\right] + 2D\sqrt{\log N(\delta, \mathcal{F}, \|.\|_2)} \\
&\leq 2E\left[\sup_{d_Y(f,f') \leq \delta} (Y_f - Y_{f'})\right] + 2D\sqrt{\log N(\delta/\sqrt{n}, \mathcal{F}, \|.\|_\infty)} \\
&\leq 2\delta\sqrt{n} + 4L\sqrt{n(L\sqrt{n})/\delta} \\
&\leq 2\delta\sqrt{n} + 4L^{3/2}\sqrt{n^{3/2}/\delta}
\end{aligned}
$$

- Set $\delta^{3/2} = CL^{3/2}n^{1/4}$, i.e. $\delta = C'Ln^{1/6}$ to get $\mathcal{R}_n = O(n^{-1/3})$