THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 385: Stat Models for Big Data

Lecture 6: Support Vector Machines

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Support Vector Machines

- Given training data $(x_i, y_i)_{i=1}^n \in \mathbb{R}^p \times \{-1, 1\}$, we want to minimize:

$$\min_w \frac{w^T w}{2} + C \sum_i \max(0, 1 - y_i w^T x_i)$$

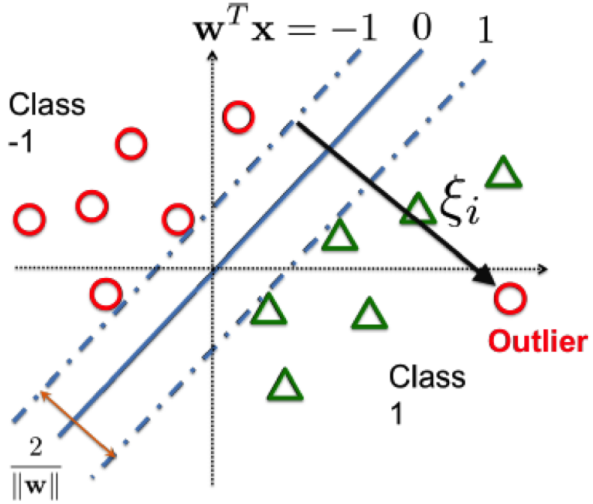


Figure 1: Courtesy Cho-Jui Hsieh's class

SGD for SVM

- Define:

$$f(w) = \frac{1}{n} \sum_i \left(\underbrace{\frac{w^T w}{2} + nC \max(0, 1 - y_i w^T x_i)}_{f_i(w)} \right)$$

- For $t = 1 \ldots$
  - Pick j uniformly at random.
  - Compute $\nabla f_j(w)$
  - Update $w = w - \eta_t \nabla f_j(w)$

SGD for SVM

- In this case, the hinge loss is not differentiable.
- A subgradient of the hinge loss $\max(0, 1 - y_i w^T x_i)$

$$\begin{cases} -y_i x_i & \text{if } 1 - y_i w^T x_i > 0 \\ 0 & \text{if } 1 - y_i w^T x_i < 0 \\ 0 & \text{if } 1 - y_i w^T x_i = 0 \end{cases}$$

SGD for SVM

- For $t = 1 \ldots$
  - Pick j uniformly at random.
  - If $y_j w^T x_j < 1$
    - $w_{t+1} = w_t(1 - \eta_t) + \eta_t C n y_i x_i$
  - Else update $w_{t+1} = w_t(1 - \eta_t)$
    - If you store w as a scalar, vector pair (γ, v) such that $w = \gamma v$, then just updating γ leads to $O(1)$ computation.
- This is in "Pegasos: primal estimated subgradient solver for SVM", ICML 2007, Shalev-Schwartz et al.

Support Vector Machines – Dual

- Given training data $(x_i, y_i)_{i=1}^n \in \mathbb{R}^p \times \{-1, 1\}$, we want to minimize:

$$\min_w \frac{w^T w}{2} + C \sum_i \underbrace{\max(0, 1 - y_i w^T x_i)}_{\xi_i}$$

where

$$\xi_i = \max(0, 1 - y_i w^T x_i) \Rightarrow \xi_i \geq 0, \xi_i \geq 1 - y_i w^T x_i$$

Dual

- Remember the primal problem?

$$\min_{w,\xi} \frac{w^T w}{2} + C \sum_i \xi_i$$
$$s.t. \quad y_i w^T x_i - 1 + \xi_i \geq 0, \xi_i \geq 0, i = 1, \ldots, n$$

- Add lagrange multipliers:

$$\min_{w,\xi} \max_{\alpha \geq 0, \beta \geq 0} \frac{w^T w}{2} + C \sum_i \xi_i - \sum_i \alpha_i (y_i w^T x_i - 1 + \xi_i) - \sum_i \beta_i \xi_i$$

- Under Slater's condition, exchanging min and max does not change the optimal solution.

Dual

- The dual is:

$$\max_{\alpha \geq 0, \beta \geq 0} \min_{w, \xi} \frac{w^T w}{2} + C \sum_i \xi_i - \sum_i \alpha_i (y_i w^T x_i - 1 + \xi_i) - \sum_i \beta_i \xi_i$$

- Differentiate w.r.t w.

$$w^* = \sum_i \alpha_i^* y_i x_i$$

- Differentiate w.r.t ξ_i.

$$C = \alpha_i + \beta_i$$

- Substituting

$$\max_{0 \leq \alpha \leq C} -\frac{1}{2} \sum_{ij} \alpha_i \alpha_j y_i y_j x_i^T x_j + \sum_i \alpha_i$$

SVM: the dual problem

- The dual of SVM is given by:

$$\min_{\alpha} \frac{1}{2} \alpha^T Q \alpha - \sum_i \alpha_i$$

$$\text{s.t.} \alpha_i \in [0, C]$$

  Where $Q_{ij} = y_i y_j x_i^T x_j$.

- The primal solution can be written in terms of the dual solution as:

$$w^* = \sum_i y_i \alpha_i^* x_i$$

Stochastic Dual coordinate descent

- Consider the one variable problem:

$$f(\alpha + \delta e_i) = \frac{1}{2}(\alpha + \delta e_i)^T Q(\alpha + \delta e_i) - \sum_i \alpha_i - \delta$$
$$= \frac{1}{2}\alpha^T Q\alpha + \delta\alpha^T Qe_i + \delta^2 \frac{Q_{ii}}{2} - \sum_i \alpha_i - \delta$$

- Set the gradient to zero:

$$(Q\alpha)_i + Q_{ii}\delta^* - 1 = 0 \rightarrow \delta^* = \frac{1 - (Q\alpha)_i}{Q_{ii}}$$

- But we have the constraint $0 \leq \alpha_i + \delta \leq C$, so we have:

$$\alpha_i + \delta^* = \begin{cases} \alpha_i + \dfrac{1 - (Q\alpha)_i}{Q_{ii}} & \text{If } \alpha_i + \delta \in [0, C] \\ 0 & \text{If } \alpha_i + \delta < 0 \\ C & \text{If } \alpha_i + \delta > C \end{cases}$$

Stochastic Dual coordinate descent

- For $t = 1 \ldots$
  - Pick a coordinate i at random.
  - Compute
$$\delta^* = \arg \min_{0 < \alpha_i + \delta < C} f(\alpha + \delta e_i)$$
  - Update $\alpha_i = \alpha_i + \delta^*$
  - Update $w = w + \delta^* y_i x_i$ (time complexity $O(nnz(x_i))$
  - After convergence this gives $w^* = \sum_i \alpha_i^* y_i x_i$

Fast computation

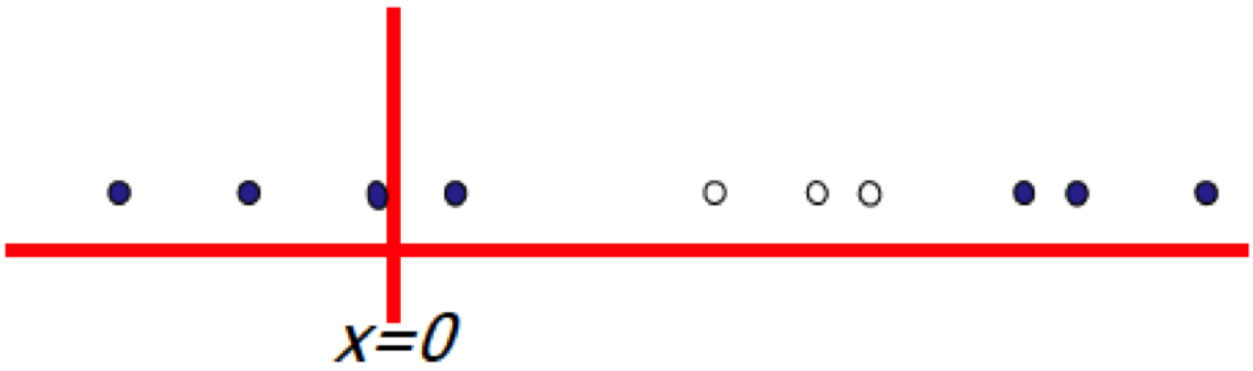
- Main computational bottleneck $Q\alpha$
- Write $Q = \underbrace{\text{diag}(y)X}_{R}\underbrace{X^T\text{diag}(y)}_{R^T}$
- Note that:

$$(Q\alpha)_i = R_i \underbrace{R^T\alpha}_{w} = y_i x_i^T w$$

- If you maintain w through the steps, computational complexity becomes $O(nnz(x_i))$
- After each $\alpha_i \leftarrow \alpha_i + \delta^* e_i$ update, you have:

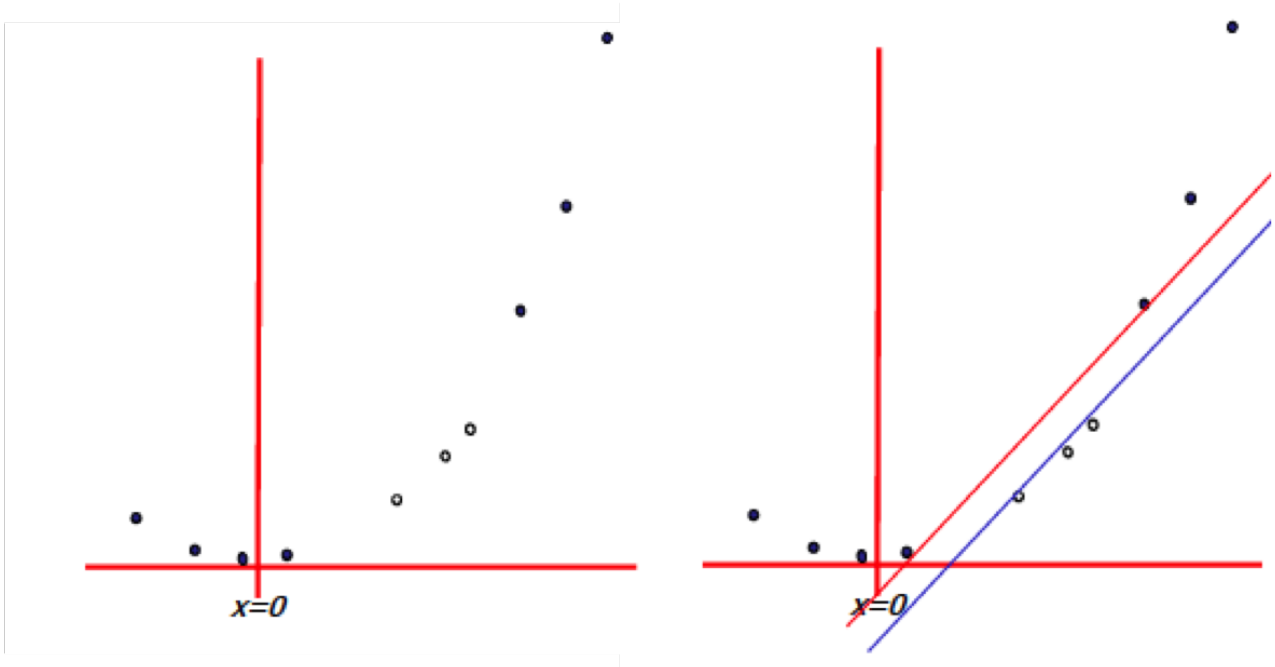
$$w \leftarrow w + \delta^* Q e_i = w + \delta^* Q(:, i)$$

SVM: the kernel trick



- How do we use an SVM here?
- What if we use more than one dimensions?
- Say $z = (x, x^2)$

SVM: the kernel trick



- So using a different mapping to a higher dimensional space helped.
- So if I want to map my features to a quadratic space, I will have coefficients: $(1, x_1, x_2, \ldots, x_1^2, x_2^2, \ldots, x_1x_2, x_1x_3 \ldots,)$
- So a total of $O(p^2)$ terms. If it is a cubic, then $O(p^3)$ terms. Wow! storage increases exponentially with the dimensionality.

SVM: the kernel trick

- So in a nutshell:

$$\min_{\alpha} \frac{1}{2}\alpha^T Q\alpha - \sum_k \alpha_k$$

$$s.t. \quad 0 \leq \alpha \leq C$$

where

$$Q_{ij} = y_i y_j \phi(x_i)^T \phi(x_j)$$

SVM: the kernel trick

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- Building Q needs $n^2 m^2$ time, where m is the number of dimensions of the projection.

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- $w = \sum_{k:\alpha_k > 0} \alpha_k y_k \phi(x_k)$. This will take $O(nm)$ time.

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- Building Q needs $n^2 m^2$ time, where m is the number of dimensions of the projection.
- $w = \sum_{k:\alpha_k > 0} \alpha_k y_k \phi(x_k)$. This will take $O(nm)$ time.
- Classification rule for datapoint x:

$$\text{Predict sign}(w^T \phi(x))$$

See next slide.

SVM: the kernel trick

- But, how do you predict the class of a new example?
  - You would need to compute $w^T \phi(x)$, where $\phi(x)$ is the high dimensional mapping.
  - This is proportional to the length of $\phi(x)$
  - But remember the form of w?
  - $w^T \phi(x) = \sum_i \alpha_i y_i \phi(x_i)^T \phi(x) = \sum_i \alpha_i y_i K(x_i, x)$

SVM: the kernel trick

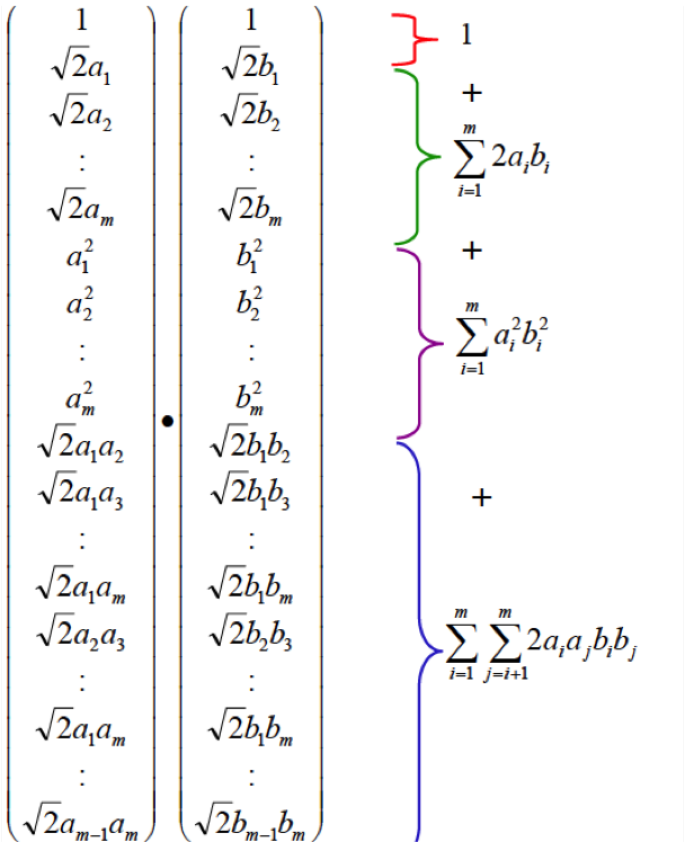

Figure 2: Courtesy: Andrew Moore's lecture slides

- But this dot product is the same as $(a^T b + 1)^2$!
- Same for cubic maps. So instead of doing $O(p^k)$ computation to compute a dot product in degree k polynomial, you can compute it

Acknowledgment

Cho-Jui Hsieh's class notes at UC Davis. Andrew Moore's notes on SVM.