Appendix for “Mean Field for the Stochastic Blockmodel: Optimization Landscape and Convergence Issues”

Abstract

This supplementary article contains an appendix to our paper “Mean Field for the Stochastic Blockmodel: Optimization Landscape and Convergence Issues”, providing derivation of stationarity equations for the mean field log-likelihood and the proofs of our main results.

1 The Variational principle and mean field

We start with the following simple observation:

\[
\log P(A; B, \pi) = \log \sum_Z P(A, Z; B, \pi) = \log \left( \sum_Z \frac{P(A, Z; B, \pi)}{\psi(Z)} \psi(Z) \right) \geq \sum_Z \log \left( \frac{P(A, Z; B, \pi)}{\psi(Z)} \right) \psi(Z) \quad \forall \psi \text{ prob. on } Z.
\]

(A.1)

In fact, equality holds for \( \psi^* (Z) = P(Z|A; B, \pi) \). Therefore, if \( \Psi \) denotes the set of all probability measures on \( Z \), then

\[
\log P(A; B, \pi) = \max_{\psi \in \Psi} \sum_Z \log \left( \frac{P(A, Z; B, \pi)}{\psi(Z)} \right) \psi(Z).
\]

(A.1)

The crucial idea from variational inference is to replace the set \( \Psi \) above by some easy-to-deal-with subclass \( \Psi_0 \) to get a lower bound on the log-likelihood.

\[
\log P(A; B, \pi) \geq \max_{\psi \in \Psi_0} \sum_Z \log \left( \frac{P(A, Z; B, \pi)}{\psi(Z)} \right) \psi(Z).
\]

(A.2)

Also the optimal \( \psi^*_* \in \Psi_0 \) is a potential candidate for an estimate of \( P(Z|A; B, \pi) \). Estimating \( P(Z|A; B, \pi) \) is profitable since then we can obtain an estimate of the community membership.
matrix by setting \(Z_{ia} = 1\) for the \(i\)th agent where
\[
a = \arg \max_b P(Z_{ib} = 1|A; B, \pi).
\] (A.3)

The goal now has become optimizing the lower bound in (A.2).

2 Derivation of stationarity equations

Every stationary point \(\theta = (\psi, p, q)\) of the mean field log-likelihood satisfies \(\nabla_\theta \ell(\theta) = 0\). In particular,
\[
0 = \frac{\partial \ell}{\partial \psi_i} = 4t \sum_{j:j \neq i} (\psi_j - \frac{1}{2})(A_{ij} - \lambda) - \log\left(\frac{\psi_i}{1 - \psi_i}\right)
\]
\[
0 = \frac{\partial \ell}{\partial p} = \frac{1}{2} \sum_{i,j:i \neq j} (\psi_i \psi_j + (1 - \psi_i)(1 - \psi_j))(A_{ij}\left(\frac{1}{p} + \frac{1}{1-p}\right) - \frac{1}{1-p})
\]
\[
0 = \frac{\partial \ell}{\partial q} = \frac{1}{2} \sum_{i,j:i \neq j} (\psi_i(1 - \psi_j) + (1 - \psi_i)\psi_j)(A_{ij}\left(\frac{1}{q} + \frac{1}{1-q}\right) - \frac{1}{1-q})
\] (A.4)

Therefore
\[
\frac{\partial^2 \ell}{\partial \psi_i \partial \psi_j} = 4t(A_{ij} - \lambda)(1 - \delta_{ij}) - \frac{1}{\psi_i(1 - \psi_i)} \delta_{ij}
\]

\[
\frac{\partial^2 \ell}{\partial \psi_i \partial p} = \frac{1}{2} \sum_{j:j \neq i} \left(\frac{1}{2} - \psi_j\right)\left(A_{ij}\left(\frac{1}{p} + \frac{1}{1-p}\right) - \frac{1}{1-p}\right)
\]

\[
\frac{\partial^2 \ell}{\partial \psi_i \partial q} = \frac{1}{2} \sum_{j:j \neq i} \left(\psi_i - \frac{1}{2}\right)\left(A_{ij}\left(\frac{1}{q} + \frac{1}{1-q}\right) - \frac{1}{1-q}\right)
\]

\[
\frac{\partial^2 \ell}{\partial p^2} = \frac{1}{2} \sum_{i,j:i \neq j} (\psi_i \psi_j + (1 - \psi_i)(1 - \psi_j))(A_{ij}\left(-\frac{1}{p^2} + \frac{1}{(1-p)^2}\right) - \frac{1}{(1-p)^2})
\]

\[
\frac{\partial^2 \ell}{\partial q^2} = \frac{1}{2} \sum_{i,j:i \neq j} (\psi_i(1 - \psi_j) + (1 - \psi_i)\psi_j)(A_{ij}\left(-\frac{1}{q^2} + \frac{1}{(1-q)^2}\right) - \frac{1}{(1-q)^2})
\]

\[
\frac{\partial^2 \ell}{\partial q \partial p} = 0.
\] (A.5)

3 Proofs of main results

Proof of Proposition 3.1 For any \(a > b > 0\), we have
\[
\frac{a-b}{a} < \log\left(\frac{a}{b}\right) < \frac{a-b}{b},
\]
which can be proved using the inequality \(\log(1 + x) < x\) for \(x > -1\), \(x \neq 0\). Therefore
\[
\frac{p-q}{p} < \log\left(\frac{p}{q}\right) < \frac{p-q}{q}, \quad \text{and} \quad \frac{p-q}{1-q} < \log\left(\frac{1-q}{1-p}\right) < \frac{p-q}{1-p}.
\]

So
\[
\frac{(p-q)(1+p-q)}{2(1-q)p} < t = \frac{1}{2}\left(\log\left(\frac{p}{q}\right) + \log\left(\frac{1-q}{1-p}\right)\right) < \frac{(p-q)(1+p+q)}{2(1-p)q},
\]

and
\[
q = \frac{\frac{p-q}{1-q}}{\frac{p-q}{p} + \frac{p-q}{1-q}} < \lambda = \frac{\log\left(\frac{1-q}{1-p}\right) + \log\left(\frac{1-q}{1-p}\right)}{\frac{p-q}{p} + \frac{p-q}{1-p}} < \frac{\frac{p-q}{1-p}}{\frac{p-q}{p} + \frac{p-q}{1-p}} = p.
\]

\(\square\)
3.1 Proofs of results in Section 3.1

\textbf{Proof of Proposition 3.2.} That \( \psi = \frac{1}{2} \mathbf{1} \) is a stationary point is obvious from the stationarity equations (A.4). The eigenvalues of \(-4I + 4tM\), the Hessian at \( \frac{1}{2} \mathbf{1} \), are \( h_i = -4 + 4t\nu_i \). We have \( \nu_1 = n\alpha_+ - \langle p - \lambda \rangle = \Theta(n) \), and hence so is \( h_1 \). Also, \( p - \lambda > 0 \), so that \( \nu_3 < 0 \), and hence \( h_3 < 0 \). Thus we have two eigenvalues of the opposite sign. \qed

\textbf{Proof of Theorem 3.3.} From (5), we have
\[
\psi^{(s+1)} = g(na_1^{(s)}) = g(na_1^{(s)} + \delta_1^{(s)}) \text{,}
\]
where \( |\delta_1^{(s)}| = O(\exp(-n|a_1^{(s)}|)) \), where we have used the fact that
\[g(nx + y) - g(nx) = g(nx)g(nx + y)(e^y - 1)\exp(-(nx + y)).\]

Writing as a vector, we have
\[
\psi^{(s+1)} = g(na_1^{(s)}) \mathbf{1}_{c_1} + g(na_1^{(s)}) \mathbf{1}_{c_2} + \delta^{(s)}, \quad (A.6)
\]
where \( \|\delta^{(s)}\|_\infty = \max_i |\delta_i^{(s)}| = O(\exp(-n \min\{|a_1^{(s)}|, |a_1^{(s)}|\})) \). Note that by our assumption, \( \|\delta^{(0)}\|_\infty = O(\exp(-n \min\{|a_1^{(s)}|, |a_1^{(s)}|\})) = o(1) \). Now
\[
\zeta_1^{(s+1)} = \frac{\langle \psi^{(s+1)}, u_1 \rangle}{n} = \frac{g(na_1^{(s)}) + g(na_1^{(s)})}{2} + O(\|\delta^{(s)}\|_\infty),
\]
and
\[
\zeta_2^{(s+1)} = \frac{\langle \psi^{(s+1)}, u_2 \rangle}{n} = \frac{g(na_1^{(s)}) - g(na_1^{(s)})}{2} + O(\|\delta^{(s)}\|_\infty).
\]

Note that \( g(na_1^{(s)}) = 1_{\{a_1^{(s)} > 0\}} + O(\|\delta^{(s)}\|_\infty) \). Now, using (A.6), we have
\[
\|\psi^{(s+1)} - \ell(\psi^{(0)})\|_2^2 \leq \frac{2\|g(na_1^{(s)}) - 1_{\{a_1^{(s)} > 0\}}\|_2^2 + \|g(na_1^{(s)}) - 1_{\{a_1^{(s)} > 0\}}\|_2^2 + \|\delta^{(s)}\|_2^2}{n} \leq |g(na_1^{(s)}) - 1_{\{a_1^{(s)} > 0\}}|^2 + |\delta^{(s)}|_2^2 + O(\|\delta^{(s)}\|_\infty^2).
\]

From the above representation and our assumption on \( n|a_1^{(0)}| \), the bound for \( s = 1 \) follows. We will now consider the four different cases of different signs of \( a_1^{(s)} \).

\textbf{Case 1:} \( a_1^{(s)} > 0 \), \( a_1^{(s)} > 0 \). In this case \( g(na_1^{(s)}) = g(na_1^{(s)}) = 1 + O(\|\delta^{(s)}\|_\infty) \), so that
\[
(\zeta_1^{(s+1)}, \zeta_2^{(s+1)}) = (1, 0) + O(\|\delta^{(s)}\|_\infty).
\]

This implies
\[a_1^{(s+1)} = 2t\alpha_+ + O(\|\delta^{(s)}\|_\infty).\]

If \( \alpha_+ > 0 \), \( a_1^{(s+1)} \) have the same sign as \( a_1^{(s)} \). Otherwise, if \( \alpha_+ < 0 \), both of them become negative (and we thus have to go to Case 2 below). Note that, here and in the subsequent cases, we are using that fact that \( \|\delta^{(s)}\|_\infty = o(1) \), for \( s = 0 \), by our assumption and it stays the same for \( s \geq 1 \) because of relations like the above (that is \( a_1^{(1)} = -2t\alpha_+ + o(1) \), so that \( \|\delta^{(1)}\|_\infty = \exp(-n \min\{|a_1^{(1)}|, |a_1^{(1)}|\}) = O(\exp(-Cn\alpha_+)) = o(1) \), and so on).
We conclude that, if $\psi \in A$.

**Case 1:** $a^{(s)}_1 < 0, a^{(s)}_{-1} < 0$. In this case $1 - g(na^{(s)}_1) = 1 - g(na^{(s)}_{-1}) = 1 + O(\|\delta^{(s)}\|_\infty)$, so that

$$(\zeta_1^{(s+1)}, \zeta_2^{(s+1)}) = (0, 0) + O(\|\delta^{(s)}\|_\infty).$$

This implies

$$a^{(s+1)}_\pm = -2t\alpha_\pm + O(\|\delta^{(s)}\|_\infty).$$

If $\alpha_+ > 0$, $a^{(s+1)}_\pm$ have the same sign as $a^{(s)}_\pm$. Otherwise, if $\alpha_+ < 0$, both of them become positive (and we thus have to go to Case 1 above).

**Case 2:** $a^{(s)}_1 > 0, a^{(s)}_{-1} < 0$. In this case $g(na^{(s)}_1) = 1 - g(na^{(s)}_{-1}) = 1 + O(\|\delta^{(s)}\|_\infty)$, so that

$$(\zeta_1^{(s+1)}, \zeta_2^{(s+1)}) = (0, 0) + O(\|\delta^{(s)}\|_\infty).$$

This implies

$$a^{(s+1)}_\pm = +2t\alpha_\pm + O(\|\delta^{(s)}\|_\infty).$$

Since $\alpha_- > 0$, $a^{(s+1)}_\pm$ have the same sign as $a^{(s)}_\pm$.

**Case 3:** $a^{(s)}_1 < 0, a^{(s)}_{-1} < 0$. In this case $g(na^{(s)}_1) = 1 - g(na^{(s)}_{-1}) = 1 + O(\|\delta^{(s)}\|_\infty)$, so that

$$(\zeta_1^{(s+1)}, \zeta_2^{(s+1)}) = (0, 0) + O(\|\delta^{(s)}\|_\infty).$$

This implies

$$a^{(s+1)}_\pm = +2t\alpha_\pm + O(\|\delta^{(s)}\|_\infty).$$

Since $\alpha_- > 0$, $a^{(s+1)}_\pm$ have the same sign as $a^{(s)}_\pm$.

Note that, in the case $\alpha_+ = 0$, $a^{(s)}_\pm = \pm \alpha_\pm$ have opposite signs and we land in Cases 3 or 4.

We conclude that, if $\alpha_+ \geq 0$, then we stay in the same case where we began, and otherwise if $\alpha_+ < 0$ we have a cycling behavior between Cases 1 and 2. Now the desired conclusion follows from the bound (A.7).

In the proof above, we can allow sparser graphs, with $p, q \gg \frac{1}{n}$. More explicitly, let $p = \rho_n a, q = \rho_n b$, with $a > b > 0$ and $\rho_n \gg \frac{1}{n}$. Then, $t = \Omega(1)$, and $\alpha_+ \leq p - q = \rho_n (a - b), \alpha_- = (p - q)/2 = \rho_n (a - b)/2$. So, we do have $n|\alpha_\pm| \to \infty.$

**Proof of Theorem 3.4.** We begin by noting that $\hat{M} - M = A - E(A|Z)$. For the first iteration, we rewrite the sample iterations (7) as

$$\hat{\xi}^{(1)} = 4t\hat{M} \left( \psi^{(0)} - \frac{1}{2} \mathbf{1} \right) + 4t(\hat{M} - M) \left( \psi^{(0)} - \frac{1}{2} \mathbf{1} \right).$$

Therefore, similar to the population case, we have

$$\hat{\psi}_i^{(1)} = g(na^{(0)}_i + b^{(0)}_i + nE^{(0)}_i).$$

Note that

$$\psi^{(0)}_i = \frac{4t}{n} \sum_{j \neq i} (A_{ij} - E(A_{ij}|Z_i, Z_j))(\psi^{(0)}_j - \frac{1}{2}).$$

Assume that $\psi^{(0)}$ is independent of $A$. Since our probability statements will be with respect to the randomness in $A$, we may assume that $\psi^{(0)}$ is fixed. Let $Y_{ij} = (A_{ij} - E(A_{ij}))\psi^{(0)} - \frac{1}{2}$. Then the $Y_{ij}$ are independent random variables for $j \neq i$, and $E(Y_{ij}) = 0$. Also, $|Y_{ij}| \leq |\psi^{(0)} - \frac{1}{2}| \leq$
All in all, we have

$$\text{Proof of Proposition 3.6.}$$

More explicitly,

$$H_{n} \cap H_{n}^\gamma \cap [0, 1]^n,$$

All in all, we have

$$\mathcal{M}(S_1) \geq \lim_{\gamma \uparrow 1} \mathcal{M}(H_{n}^\gamma \cap H_{n}^\gamma \cap [0, 1]^n).$$

$$\text{Proofs of results in Section 3.2.}$$

$$\text{Proof of Proposition 3.6.}$$ That the described point is a stationary point is easy to verify, because of the presence of the $$\psi_j(0)$$ terms in the stationarity equations $$\mathcal{A}_k \mathcal{A}_k \mathcal{A}_k \mathcal{A}_k$$. Now, from $$\mathcal{A}_k \mathcal{A}_k \mathcal{A}_k \mathcal{A}_k$$ we see that the Hessian matrix at $$\mathcal{A}_k \mathcal{A}_k \mathcal{A}_k \mathcal{A}_k$$ is given by

$$H = \begin{pmatrix} -4 I & 0 & 0 & 0 \\ 0 & -\frac{n(n-1)}{4(n-a)} & 0 & 0 \\ 0 & 0 & -\frac{n(n-1)}{4(n-a)} & 0 \\ 0 & 0 & 0 & -\frac{n(n-1)}{4(n-a)} \end{pmatrix},$$

where $$\hat{a} = \frac{1}{n(n-1)}$$. Clearly, $$H$$ is negative definite. This completes the proof.
Proof of Lemma 3.1. First note that conditioning on the true labels \( Z, \mathbb{E}(A|Z) = P - pI \). For the update of \( p^{(1)} \), we have

\[
p^{(1)} = \frac{\psi^T (P - pI) \psi + (1 - \psi)^T (P - pI) (1 - \psi)}{\psi^T (J - I) \psi + (1 - \psi)^T (J - I) (1 - \psi)}
\]

where the first term can be written as

\[
\psi^T \left( \frac{p+q}{2} u_1 u_1^T + \frac{p-q}{2} u_2 u_2^T - pI \right) \psi + (1 - \psi)^T \left( \frac{p+q}{2} u_1 u_1^T + \frac{p-q}{2} u_2 u_2^T - pI \right) (1 - \psi)
\]

\[
= \frac{p+q}{2} n^2 (\zeta_1^2 + (1 - \zeta_1)^2) + n^2 (p-q) \zeta_2^2 - px
\]

\[
= \frac{p+q}{2} + (p-q) (\zeta_2^2 - x/2n^2)
\]

where \( x = \psi^T \psi + (1 - \psi)^T (1 - \psi) \geq n^2/4 \). The second term can be bounded by noting \( \mathbb{E}(\psi^T (A - \mathbb{E}(A|Z)) \psi) = 0 \) and \( \text{Var}(\psi^T (A - \mathbb{E}(A|Z)) \psi) \leq 2n(n-1)p \). By Chebyshev’s inequality, \( \psi^T (A - \mathbb{E}(A|Z)) \psi = \mathcal{O}(\sqrt{\rho_n n}) \).

This is because:

\[
E_{\psi,A}[\psi^T (A - \mathbb{E}(A|Z)) \psi] = E_{\psi}E_A[\psi^T (A - \mathbb{E}(A|Z)) \psi] = 0,
\]

and,

\[
\text{Var}_{\psi,A}[\psi^T (A - \mathbb{E}(A|Z)) \psi] = E \text{Var}(\psi^T (A - \mathbb{E}(A|Z)) \psi) + \text{Var}(E(\psi^T (A - \mathbb{E}(A|Z)) \psi) \psi)
\]

\[
= 4E \sum_{i<j} \psi_i \psi_j \text{Var}(A_{ij}) \leq 2n(n-1)p
\]

Similarly for \((1 - \psi)^T (A - \mathbb{E}(A)) (1 - \psi)\) and

\[
\psi^T (J - I) \psi + (1 - \psi)^T (J - I) (1 - \psi)
\]

\[
= \left( \sum_i \psi_i \right)^2 + \left( n - \sum_i \psi_i \right)^2 - \psi^T \psi - (1 - \psi)^T (1 - \psi)
\]

\[
\geq n^2/2 - 2n.
\]

since the first two terms are minimized at \( \sum_i \psi_i = n/2 \).

The update rule for \( q^{(1)} \) is proved analogously.

Proof of Proposition 3.7. Let \( \psi = \zeta_1 u_1 + \zeta_2 u_2 + w, w \in \text{span} \{ u_1, u_2 \}^\perp \), be a stationary point. We will consider the population version of all the updates and replace \( A \) with \( \mathbb{E}(A|Z) = P - pI \) and \( \rho_n \to 0 \). By Lemma 3.1

\[
\dot{p} = \frac{p + q}{2} + \frac{(p-q)(\zeta_2^2 - x/2n^2)}{\zeta_1^2 + (1 - \zeta_1)^2 - x/n^2} \cdot \frac{\zeta_1}{\zeta_1^2}
\]

\[
\dot{q} = \frac{p + q}{2} - \frac{(p-q)(\zeta_2^2 + y/2n^2)}{2\zeta_1(1 - \zeta_1) - y/n^2}.
\]  

(A.8)
In this case, the update equation \([4]\) becomes
\[
\xi = 4\tilde{t}(P - pI - \tilde{\lambda}(J - I))(\psi^{(s)} - \frac{1}{2}I) \\
= 4\tilde{t}n \left( \left( \zeta_1 - \frac{1}{2} \right) \frac{p + q}{2} - \tilde{\lambda} \right) u_1 + \frac{p - q}{2} \zeta_2 u_2 + 4\tilde{t}(\tilde{\lambda} - p) \left( \psi - \frac{1}{2}I \right) \\
:= n\tilde{a} + \tilde{b} 
\]
(A.9)
where \(\tilde{\lambda}\) and \(\tilde{t}\) are defined in terms of \(\tilde{p}\) and \(\tilde{q}\). Since \(\psi\) is a stationary point, the above update gives \(\psi = g(\xi)\).

We consider the following cases.

**Case 1:** \(\zeta_2 = \Omega(1)\). Since \(\zeta_1(1 - \zeta_1) \geq \zeta_2\), it is easy to see (A.8) implies \(\tilde{p} > \frac{p + q}{2} > \tilde{q}\), thus \(\tilde{p} - \tilde{q} = \Omega(\rho_n), \tilde{t} = \Omega(1), \tilde{p} < \tilde{\lambda} < \tilde{q}\). It follows then \(\tilde{b}_i = O(\rho_n)\), and \(|\tilde{a}_i| = \Omega(\rho_n)\) for \(i \in C_1\) or \(i \in C_2\) (or both). In any of these cases, \(\|w\| = O(\rho_n\sqrt{n}) = o(\sqrt{n})\).

**Case 2:** \(\zeta_2 = o(1)\). Note that \(\psi^T(1 - \psi) \geq 0\) implies \(\zeta_1(1 - \zeta_1) - \frac{\|w\|^2}{n} \geq \zeta_2\). If \(\|w\|^2 = o(n)\) we are done. If \(\|w\|^2 = O(n), \zeta_1(1 - \zeta_1) = \Omega(1)\). In this case, \(\tilde{p} = \frac{p + q}{2} + O(\rho_n\zeta_2)\), similarly for \(\tilde{q}\). It follows then \(\tilde{t} = O(\zeta_2) = o(1), \tilde{\lambda} = \frac{p + q}{2} + O(\rho_n)\) (we defer the details to (A.12)–(A.16)). Also note that \(\tilde{b}_i = O(\rho_n\zeta_2)\). When \(n|\tilde{a}_i| \gg \tilde{b}_i, g(\xi) = g(n\tilde{a}_i) + o(1)\). Since \(g(n\tilde{a}) \in \text{span}\{u_1, u_2\}\), this implies \(\|w\| = o(\sqrt{n})\). When \(n|\tilde{a}_i| \gg \tilde{b}_i, \xi = o(1)\), so we have \(\|w\| = o(\sqrt{n})\) again.

**Proof of Lemma 3.3:** Let \(a = (p + q)/2\). By \([5]\), define \(\kappa_1 := 4t \left( \zeta_1 - \frac{1}{2} \right)(a - \lambda)\) and \(\kappa_2 = 4t\zeta_2 \frac{p - q}{2}\). Consider the initial distribution \(\psi^{(0)} \sim \nu \mu\) where \(\nu\) is a distribution supported on \((0, 1)\) with mean \(\mu\). Note that we have the following:
\[
\zeta_1 = \frac{\psi^T_1}{n} = \mu + O_P(1/\sqrt{n}), \\
\zeta_2 = \frac{\psi^T u_2}{n} = O_P(1/\sqrt{n}). 
\]
(A.10)
Now using \([10]\), recall:
\[
p^{(1)} = \frac{p + q}{2} + \frac{(p - q)(\zeta_2^2 - x/2n^2)}{\zeta_1^2 + (1 - \zeta_1)^2 - x/n^2} + O_P(\sqrt{n}/n) \\
q^{(1)} = \frac{p + q}{2} - \frac{(p - q)(\zeta_2^2 + y/2n^2)}{2\zeta_1^2(1 - \zeta_1) - y/n^2} - O_P(\sqrt{n}/n) 
\]
(A.11)
This gives:
\[
\epsilon_1 = \epsilon'_1 + O_P\left( \frac{\sqrt{\rho_n}}{n} \right) = O_P\left( \frac{\rho_n}{n} \right) + O_P\left( \sqrt{\frac{\rho_n}{n}} \right) = O_P\left( \sqrt{\rho_n/n} \right), \\
\epsilon_2 = \epsilon'_2 + O_P\left( \frac{\sqrt{\rho_n}}{n} \right) = O_P\left( \frac{\sqrt{\rho_n}}{n} \right). 
\]
We will use the following logarithmic inequalities for \(a > \epsilon > 0\):
\[
\frac{2\epsilon}{a + \epsilon} \leq \log \frac{a + \epsilon}{a - \epsilon} \leq \frac{2\epsilon}{a - \epsilon}. 
\]
(A.12)
Now we have
\[
t = \frac{1}{2} \left( \log \left( \frac{a + \epsilon_1}{a - \epsilon_2} \right) + \log \left( \frac{1 - a + \epsilon_2}{1 - a - \epsilon_1} \right) \right).
\]
\[
2t \geq \frac{\epsilon_1 + \epsilon_2}{a + \epsilon_1} + \frac{\epsilon_1 + \epsilon_2}{1 - a + \epsilon_2} \geq \frac{\epsilon_1 + \epsilon_2}{(a + \epsilon_1)(1 - a + \epsilon_2)}.
\]
\[
2t \leq \frac{\epsilon_1 + \epsilon_2}{(a - \epsilon_2)(1 - a - \epsilon_1)}.
\] (A.13)

For \( \lambda \), if \( \epsilon_1 + \epsilon_2 \geq 0 \), we have
\[
\lambda = \frac{\log \frac{1 - q^{(1)}}{1 - p^{(1)}}}{\log \frac{1 - q^{(1)}}{1 - p^{(1)}}} \leq \frac{\epsilon_1 + \epsilon_2}{1 - a - \epsilon_1} \left( \frac{\epsilon_1 + \epsilon_2 + \epsilon_1 + \epsilon_2}{a + \epsilon_1} + \frac{\epsilon_1 + \epsilon_2}{1 - a - \epsilon_1} \right) = a + \epsilon_1.
\] (A.14)
\[
\lambda \geq \frac{\epsilon_1 + \epsilon_2}{1 - a + \epsilon_2} \left( \frac{\epsilon_1 + \epsilon_2 + \epsilon_1 + \epsilon_2}{a - \epsilon_2} + \frac{\epsilon_1 + \epsilon_2}{1 - a + \epsilon_2} \right) = a - \epsilon_2.
\] (A.15)

If \( \epsilon_1 + \epsilon_2 \leq 0 \),
\[
\lambda = \frac{\log \frac{1 - q^{(1)}}{1 - p^{(1)}}}{\log \frac{1 - q^{(1)}}{1 - p^{(1)}}} \geq \frac{\epsilon_1 + \epsilon_2}{1 - a - \epsilon_1} \left( \frac{\epsilon_1 + \epsilon_2 + \epsilon_1 + \epsilon_2}{a + \epsilon_1} + \frac{\epsilon_1 + \epsilon_2}{1 - a - \epsilon_1} \right) = a + \epsilon_1.
\] (A.16)
\[
\lambda \leq \frac{\epsilon_1 + \epsilon_2}{1 - a + \epsilon_2} \left( \frac{\epsilon_1 + \epsilon_2 + \epsilon_1 + \epsilon_2}{a - \epsilon_2} + \frac{\epsilon_1 + \epsilon_2}{1 - a + \epsilon_2} \right) = a - \epsilon_2.
\]

Now we are ready to estimate \( \xi_i \). We define:
\[
\kappa_1 = 4t(\zeta_1 - 1/2)(a - \lambda) \leq \frac{2(\epsilon_1 + \epsilon_2)}{(a - \epsilon_2)(1 - a - \epsilon_1)} \left( \frac{1}{2} + O_P(1/\sqrt{n}) \right) \max(|\epsilon_1|, |\epsilon_2|)
\]
\[
\leq 4 \max\{\epsilon_1^2, \epsilon_2^2\} \frac{(p - q)}{a(1 - a) + O_P(\sqrt{\rho_n}/n)} \left( \frac{1}{2} + O_P(1/\sqrt{n}) \right) = O_P(1/n^2),
\]
\[
\kappa_2 = 4t\zeta_2 \frac{(p - q)}{2} \leq \frac{2(\epsilon_1 + \epsilon_2)}{(a - \epsilon_2)(1 - a - \epsilon_1)} (p - q)O_P \left( \frac{1}{\sqrt{n}} \right)
\]
\[
\leq \frac{4 \max\{|\epsilon_1|, |\epsilon_2|\}}{a(1 - a) + O_P(\sqrt{\rho_n}/n)} (p - q)O_P(1/\sqrt{n}) = O_P(\sqrt{\rho_n}/n^{3/2}).
\] (A.17)

From (\( \xi_i^{(1)} = n(\kappa_1 + \sigma, \kappa_2) + O_P(\sqrt{\rho_n}/n) = O_P(\sqrt{\rho_n}/n) \)

since \( t = O_P(1/(n\sqrt{\rho_n})) \) by (A.13).

Now applying the update for \( \psi \), we have:
\[
\psi^{(1)}_i = g \left( O_P(\sqrt{\rho_n}/n) \right) = 1/2 + O_P(\sqrt{\rho_n}/n).
\]

\[
\square
\]

**Proof of Lemma 3.3** In this setting, we write \( p^{(1)}, q^{(1)} \) as follows:
\[
p^{(1)} = p - (p - q)\frac{\zeta_i^2 + (1 - \zeta_i)^2}{\zeta_i^2 + (1 - \zeta_i)^2} - \frac{\zeta_i^2}{2} + O_P(\sqrt{\rho_n}/n),
\]
\[
q^{(1)} = q + (p - q)\frac{\zeta_i(1 - \zeta_i) - \zeta_i^2 - y/n^2}{2\zeta_i(1 - \zeta_i) - y/n^2} + O_P(\sqrt{\rho_n}/n).
\] (A.18)

From the proof of Lemma 3.2, Equation A.11 and Equation A.18 we have: \( \epsilon_1, \epsilon_2 < \frac{p + q}{2} \).

Also note that \( \epsilon_1, \epsilon_2 = \Omega_P((p - q)\zeta_i^2 + \sqrt{\rho_n}/n) \). Hence by the same argument as in Lemma 3.2 \(|(p + q)/2 - \lambda| \leq \max(|\epsilon_1|, |\epsilon_2|) = \frac{\zeta_i^2}{2} + O_P(1/n) \) by (A.18).
Finally we see that

\[ t = \Theta \left( \frac{\epsilon_1 + \epsilon_2}{\rho} \right) = \Theta \left( \frac{(p - q)\zeta_2^2}{\rho_n} \right) \]

In addition, condition (13) implies \( \zeta_2^2 = \Omega_P(1) \), we see that \( t = \Omega_P(1) \) using (A.13).

Next, using (12) and (A.17) we have

\[ \kappa_1 + \kappa_2 = 4t \left( \frac{\mu_1 + \mu_2 - 1}{2} \left( \frac{p + q}{2} - \lambda \right) + \frac{2}{4} (\mu_1 - \mu_2)(p - q) + O_P(\rho_n/\sqrt{n}) \right), \]
\[ \kappa_1 - \kappa_2 = 4t \left( \frac{\mu_1 + \mu_2 - 1}{2} \left( \frac{p + q}{2} - \lambda \right) - \frac{2}{4} (\mu_1 - \mu_2)(p - q) + O_P(\rho_n/\sqrt{n}) \right). \]

Then condition (13) implies

\[ n^2(\kappa_1^2 - \kappa_2^2) \leq n^2t^2(p - q)^2 \left( (\mu_1 + \mu_2 - 1)^2 - (\mu_1 - \mu_2)^2 + O_P \left( \frac{\rho_n}{\sqrt{n}(p - q)} \right) \right) < 0, \]

thus \( n(\kappa_1 + \kappa_2) \) and \( n(\kappa_1 - \kappa_2) \) have opposite signs. We will now check if \( n(\kappa_1 + \sigma_i\kappa_2) \to \infty \), and it suffices to lower bound \( n(|\kappa_2| - |\kappa_1|) \). Since \( |\mu_1 - \mu_2| \geq 2|\mu_1 + \mu_2 - 1| + O_P \left( \frac{\rho_n}{\sqrt{n}(p - q)} \right) \),

\[ n(|\kappa_2| - |\kappa_1|) \geq c nt(p - q)\sigma_i|\mu_1 - \mu_2| = \sigma_i \Theta \left( |\mu_1 - \mu_2|^3 \frac{(p - q)^2}{\rho_n} \right) \]

for some constant \( c \), so as long as \( |\mu_1 - \mu_2| \geq \left( \frac{\rho_n \log n}{n(p - q)^2} \right)^{1/3} \).

Thus \( \kappa_1 + \sigma_i\kappa_2 \) is growing to infinity with an order bounded below by \( \Omega_P(\log n) \).

If \( n(\kappa_1 + \kappa_2) > 0 \), since \( \psi_i^{(1)} = g(n(\kappa_1 + \sigma_i\kappa_2) + b_i) \), we have \( \psi_i^{(1)} = 1_{c_i} + O_P(\exp(-\Omega(\log n))) \).

The case \( \kappa_1 + \kappa_2 < 0 \) is similar.