Homework Assignment 3

Due by Friday, September 20

SDS 321 Intro to Probability and Statistics

1. A gambler has a fair die and a die with all sides equal to 6 in his pocket. He selects one of the die at random; when he rolls it, it shows a 6.

   (a) (2 pt) What is the probability that it is the fair die?
   Solution: Let F = the die is fair, U = the die is unfair, X = the result of the first roll. The first step is to apply Bayes' rule, and the next one is to apply the law of total probability to determine $P(X = 6)$.

$$P(F|X=6) = \frac{P(X=6|F)P(F)}{P(X=6)} = \frac{\frac{1}{6}\cdot\frac{1}{2}}{P(X=6|F)P(F)+P(X=6|U)P(U)}$$
$$= \frac{\frac{1}{12}}{\frac{1}{12}+1\cdot\frac{1}{2}} = \frac{\frac{1}{12}}{\frac{7}{12}} = \frac{1}{7}$$

   Rubric: 1 pt for Bayes' rule, 1 pt for putting it together.

   (b) (3 pts) Suppose that he rolls the same die a second time and, again, it shows a 6. Now what is the probability that it is the fair die? *Hint: use conditional independence.*
   Solution: Let Y = the result of the second roll. This will be similar to the last problem, but with a slight trick: $P(F|X=6,Y=6) = \frac{P(X=6,Y=6|F)P(F)}{P(X=6,Y=6)}$.
   Now, we need to calculate $P(X = 6, Y = 6|F)$ and $P(X = 6, Y = 6)$. By the definition of conditional probability,

$$\begin{aligned} P(X=6,Y=6|F) &= P(X=6|Y=6,F)P(Y=6,F) \\ &= P(X=6|Y=6,F)P(Y=6|F)P(F) \end{aligned}$$

   Now we invoke conditional independence: $P(X = 6|Y = 6, F) = P(X = 6|F)$ (given that the die is fair, the probability that we roll a 6 is independent of the result of any other roll). So:

$$P(X=6,Y=6|F) = P(X=6|F)P(Y=6|F)P(F) = \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{2} = \frac{1}{72}$$

   Similarly,

$$\begin{aligned} P(X=6,Y=6) &= P(X=6|Y=6)P(Y=6) \\ &= P(X=6|Y=6,F)P(Y=6|F)P(F) \\ &\quad + P(X=6|Y=6,U)P(Y=6|U)P(U) \\ &= P(X=6|F)P(Y=6|F)P(F) + P(X=6|U)P(Y=6|U)P(U) \\ &= \frac{1}{72} + 1\cdot 1\cdot\frac{1}{2} = \frac{37}{72} \end{aligned}$$

Thus, $P(F|X=6,Y=6) = \frac{1/72}{37/72} = \frac{1}{37}$.

Rubric: 1 pt for writing Bayes rule/conditional prob correctly. 1 pt for using conditional prob or the fact that $P(66|F)$ is one of 36 cases and hence 1/36. 1 pt for denominator.

(c) (2 pts) Suppose that he rolls the same die a third time and it shows a 1. Now what is the probability that it is the fair die?

Solution: Let $Z =$ the result of the third roll. This problem is nearly identical to part (b). The following is an expression for $P(F|X=6,Y=6,Z=1)$:

$$\frac{P(Z=1|F)P(Y=6|F)P(X=6|F)P(F)}{P(Z=1|F)P(Y=6|F)P(X=6|F)P(F)+P(Z=1|U)P(Y=6|U)P(X=6|U)P(U)}$$

However, $P(Z=1|U)=0$ so the second term in the denominator is simply 0, so the whole expression simplifies to $P(F|X=6,Y=6,Z=1)=1$.

2. A true/false question is to be posed to a father-and-son team on a quiz show. Both the father and the son will independently give the correct answer with probability q. They think of the following two strategies to come up with an answer.

(a) (1 pt) Choose one of them uniformly at random and let that person answer the question. What is the probability that the team answers correctly using this strategy?

Solution: Let SC and FC be events that corresponding to the son and father being correct, each with probability q of being equal to 1 and probability $1-q$ of being equal to 0. By the strategy given, $P(\text{correct}) = \frac{1}{2}\cdot P(S=1) + \frac{1}{2}\cdot P(F=1) = q$

(b) Have them both consider the question, and then either give the common answer if they agree or, if they disagree, flip a fair coin to determine which answer to give.

i. (3 pts) What is the probability that the two answers don't match up?

Solution: We can compute the joint probabilities (since S and F are independent):

$$\begin{aligned} P(SC,FC) &= q^2 \\ P(SC,FC^c) &= q(1-q) \\ P(SC^c,FC) &= (1-q)q \\ P(SC^c,FC^c) &= (1-q)^2 \end{aligned}$$

The probability that the answers don't match up is then:
$P(\text{mismatch}) = P\left(SC,FC^c \cup SC^c,FC\right)) = 2q(1-q)$.

Rubric: 1 pt for figuring out what is the mismatch event. 1 pt for using independence. 1 pt for putting everything together.

ii. (1 pt) What is the probability that the two answers match up and the common answer is correct?

Solution: As computed in the previous part, $P(SC,FC) = q^2$.

Rubric: 1 pt for correct answer.

iii. (3 pts) What is the probability that the team answers correctly using this strategy?
Solution: This is
$P(\text{correct}) = P(\text{match and correct}) + P(\text{mismatch and correct is chosen})$.
$P(\text{mismatch and correct is chosen}) = P(\text{correct}|\text{mismatch})P(\text{mismatch})$.
When mismatched, exactly one of the father and son is correct, and we will select that person with a fair coin. As such, $P(\text{correct}|\text{mismatch}) = \frac{1}{2}$.
Therefore,

$$\begin{aligned} P(\text{correct}) &= P(SC, FC) + P(\text{correct}|\text{mismatch})P(\text{mismatch}) \\ &= q^2 + \frac{1}{2} \cdot 2q(1-q) = q^2 + q(1-q) = q^2 + q - q^2 = q \end{aligned}$$

Rubric: 1 pt for breaking up correct into two events. 1 pt for writing the probability as a sum of the two. 1 pt for putting all together.

(c) (1 pt) Which strategy should they take?
Solution: Because the probability of answering correctly is the same in either strategy, they could use either.
Rubric: 1 pt for correct answer.

3. (2+3) Think about 10 letter words where each letter is one of the 26 letters A,B,...,Z. The words don't have to mean anything. How many such words are there such that:

(a) No two consecutive letters are the same.
Solution: 26×25^9
Rubric: full score for correct. Take a point off if reasoning is right but answer is wrong.

(b) C appears at most twice.
Solution: C appears never 25^{10}. C appears exactly once 10×25^9. C appears twice $\binom{10}{2}25^8$. Add them up.
Rubric: 1/2 pt for understanding what is at most twice. 1/2 for none, 1 for exactly once, 1 for exactly twice.