Homework Assignment 5

Due Oct 18

SDS 321 Intro to Probability and Statistics

1. (5 points) A man has 10 keys, one of which opens the front door. He picks one key at random (without replacement) and stops when he has found the right key. All keys are equally likely to be picked. There are 10 possible outcomes of this "experiment", i.e. {the first key is the right key}, {the second key is the right key},. . . , {the 10^{th} key is the right key}. Prove that each of these 10 outcomes has probability 1/10. Give full score for correct reasoning. If they just say each key is equally likely without any more explanation, then take 2.5 points off. If the i-th key is the correct one, that probability is first key is incorrect times second is incorrect times....times ith key is incorrect. This is 9/10x8/9x....i/10=1/10.

2. (1 + 1+ 2 + 1 + 3 points) Consider a Bernoulli random variable X such that $P(X = 1) = p$. Calculate the following:

   (a) $E[(1 - X)^{10}]$

   (b) $E[(X - p)^4]$.

   (c) $\text{var}((X - p)^2)$.

   (d) $E[3^X 4^{1-X}]$.

   (e) $\text{var}(3^X 4^{1-X})$. Take only .5 off if the final answer is wrong but approach is correct. a) $1-p$. b) $p(1-p)^4+(1-p)p^4$, c) $p(1-p)^4+(1-p)p^4-p^2(1-p)^2$ d) $3p+4(1-p)$ e) $p(1-p)$ *** Note that $3^X 4^(1 - X)$ is just a shift of a Bernoulli by 3, and so variance remains the same.

3. (3+2) The PMF of a random variable is given by:

$$P(X = x) = \begin{cases} 2a & \text{x=0} \\ b & \text{x=1} \\ a & \text{x=2} \end{cases}$$

   We also have $E[X] = 4/5$.

   (a) What are a and b? 1 point for each equation and 1 point for the correct a and b $a + 2b = 4/5$. and $3a + b = 1$. $a = 1/5, b = 2/5$

   (b) What is $\text{var}(X)$? $var(X) = 4a + b - 16/25 = 6/5 - 16/25 = 14/25$

   3. (3pts) If a coin is tossed a sequence of times (infinitely many times), what is the probability that the first head will occur **after** the 5-th toss, given that it has not occurred in the first 2 tosses?

   In your solution let $A =$"first head after 5th toss"; and $B =$"no head in first 2 tosses"

The question asks for $p(A \mid B)$. We find $p(AB) = p(A) = (1/2)^5$ (prob of 5 tails) and $p(B) = (1/2)^2$ (prob of 2 tails), and use the definition of cond prob:

$$p(A \mid B) = \frac{p(AB)}{p(B)} = \frac{(1/2)^5}{(1/2)^2} = (1/2)^3 = 1/8.$$

[1pt] for recognizing it as cond prob; 1/2 pt (each) for $p(AB)$ and $p(B)$; [1pt] for correct definition $p(A \mid B) = p(AB)/p(B)$.

4. Charles claims he can distinguish between beer and ale 75% time. Let $p =$ Charles' probability of distinguishing the drinks. Charles' claim is $p = 0.75$. Ruth bets that he cannot and, in fact, just guesses. That is, Ruth's claim is $p = 0.5$. To settle this, a bet is made: Charles is to be given $n = 5$ small glasses, each having been filled with beer or ale, chosen by tossing a fair coin. He wins the bet if he gets 4 **or more** correct. In your answer let $X =$"number of glasses he gets correct", and use $p_1 = 0.75$ and $p_0 = 0.5$.

   (a) (3pts) Find the probability that Charles wins if his claim is right, that is, if $p = p_1$. The question is for $p(X \geq 4)$ when $p = p_1$. In that case $X \sim Binomial(n, p_1)$ and

   $$p(X \geq 4) = p(4) + p(5) = 5p_1^4(1 - p_1) + p_1^5 = p_1^4(5 - 4p_1) = 0.63$$

   using $p(4) = (5\ 4)\, p_1^4\, (1 - p_1)^1 = 5p_1^4\, (1 - p_1)$ etc. [1pt] for correct binomial, that is, with $p = p_1$; [1pt] for $p(4)$; [1pt] for $p(5)$.
   no need for the decimal answer, as expression in p_1 is okay.

   (b) (3pts) Find the probability that Charles wins if his claim is wrong, that is, if $p = p_0$.
   The question is for $p(X \geq 4)$ when $p = p_0$. Now $X \sim Binomial(n, p_0)$ and

   $$p(X \geq 4) = p(4) + p(5) = 5p_0^4(1 - p_0) + p_0^5 = 6\left(\frac{1}{2}\right)^5 = 0.1875$$

   now with $p(4) = (5\ 4)\, p_0^4\, (1 - p_0)^1 = 5p_0^4(1 - p_0) = 5p_0^5$ (since $p_0 = (1 - p_0)$:-) etc. Note that the p.m.f. for X has changed – we now use $p = p_0$. [1pt] for correct binom; [1pt] for $p(4)$; [1pt] for $p(5)$

   (c) (3pts) Assume that you believe that Charles is right with probability 0.1, and Ruth is right with probability 0.9 (that is, before you see the outcome of the bet). Given that Charles gets $X = 4$ correct, what is the probability that his claim is right? In your answer let $A =$"Charles is right" (that is, $p = p_1$), and $B =$"Charles gets 4 correct". *Hint:* $A^c =$ "Ruth is right". The question is for $p(A \mid B)$. We have $p(A) = 0.1$ and $p(A^c) = 0.9$. We know $p(B \mid A)$ as $p(4)$ from part (c) and we know $p(B \mid A^c)$ as $p(4)$ from part (d). Use Bayes' theorem to find

   $$p(A \mid B) = \frac{p(B \mid A)p(A)}{p(B \mid A)p(A) + p(B \mid A^c)p(A^c)} = \frac{.1\ p_1^4(1 - p_1)}{.1\ p_1^4(1 - p_1) + .9\ p_0^5} = .22$$

   1/2 for correct $p(A)$; 1/2 (each) for using correct $p(B \mid A)$, $p(B \mid A^c)$ and $p(A)$; [1pt] for using Bayes' theorem.