Homework Assignment 6

Due in class, **Tuesday** March 10

SDS 321 Intro to Probability and Statistics

1. (1+2+2)A continuous random variable X has PDF

$$f_X(x) = \begin{cases} c(2+x^2) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of c?

   $\int_0^1 (2+x^2)dx = \left[2x + \frac{x^3}{3}\right]_0^1 = 7/3$. So, to ensure the PDF integrates to 1, $c = 3/7$.
   Take point off for wrong integration

   (b) What is the CDF, $F_X(x)$, of X? Be sure to specify $F_X(x)$ for all values of $-\infty < X < \infty$.

   $F_X(x) = \mathbf{P}(X \le x) = \int_0^x f_X(x)dx = \begin{cases} 0 & x < 0 \\ \frac{6}{7}x + \frac{x^3}{7} & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$ Answer should be in 3 parts. Point off for any wrong part.

   (c) What is the expectation and variance of X?

$$E[X] = 3/7 \int_0^1 (2x + x^3)dx = \left[x^2 + \frac{x^4}{4}\right]_0^1 = (1 + \frac{1}{4})3/7 = \frac{15}{28}$$

$$E[X^2] = 3/7 \times \int_0^1 x^2(2+x^2)dx = 3/7\,[2/3 + 1/5] = 3/7(13/15) = \frac{13}{35}$$

$$\text{var}(X) = E[X^2] - E[X]^2 = \frac{13}{35} - (\frac{15}{28})^2$$

   Points off for wrong formula of mean/variance.

2. Based on US population statistics, we can model the height of a randomly selected man using a normal random variable with mean 69.1 inches and standard deviation 2.9 inches. We can model the height of a randomly selected woman using a normal random variable with mean 63.7 inches and standard deviation 2.7 inches.

   (a) (1+1+1+2)Write the probability of a man being between 6' and 6'2" tall, in terms of the CDF of the standard normal distribution, $\Phi(z) = \mathbf{P}(Z \le z)$ for $Z \sim N(0,1)$.

$$X = 6' = 72'' \to Z = \frac{72 - 69.1}{2.9} = 1$$

$$X = 6'2'' = 74'' \to X = \frac{74 - 69.1}{2.9} = 1.69$$

$$\mathbf{P}(72 \le X \le 74) = \mathbf{P}(1 \le Z \le 1.69) = \mathbf{P}(Z \le 1.69) - \mathbf{P}(Z \le 1) = \Phi(1.69) - \Phi(1)$$

   Wrong formula for standardization – point off. Wrong look up – point off.

(b) Using the look-up table for the standard normal table (see attached table) and your answer from [a], what is the probability of a man being between 6' and 6'2" tall?

$$\mathbf{P}(72 \leq X \leq 74) = \Phi(1.68) - \Phi(1) = 0.9545 - 0.8413 = 0.1132$$

(I haven't extrapolated between the values of 1.68 and 1.69, but if you did, even better) Wrong formula for standardization – point off. Wrong look up – point off.

(c) What is the probability of a randomly selected individual being less than 5'6" tall? You may assume that $\mathbf{P}(\text{man}) = \mathbf{P}(\text{woman}) = 0.5$.

$$\mathbf{P}(X \leq 66|F) = \mathbf{P}(Z \leq \frac{66 - 63.7}{2.7}) = \mathbf{P}(Z \leq 0.85) = 0.8023$$

$$\mathbf{P}(X \leq 66|M) = \mathbf{P}(Z \leq \frac{66 - 69.1}{2.9}) = \mathbf{P}(Z \leq -1.07) = 1-\mathbf{P}(Z < 1.07) = 1-0.8557 = 0.1443$$

$$\mathbf{P}(Z \leq 66) = \mathbf{P}(X \leq 66|F)\mathbf{P}(F)+\mathbf{P}(Z \leq 66|M)\mathbf{P}(M) = 0.5\times0.8023+0.5\times0.1443 = 0.4733$$

If they use total probability rule–one point.

(d) Given the fact that a randomly selected individual is shorter than 5'5", what is the probability that the individual is a man?

$$\mathbf{P}(M|X < 65) = \frac{\mathbf{P}(X < 65|M)\mathbf{P}(M)}{\mathbf{P}(X < 65|M)\mathbf{P}(M) + \mathbf{P}(X < 65|F)\mathbf{P}(F)} = \frac{\mathbf{P}(X < 65|M)}{\mathbf{P}(X < 65|M)\mathbf{P}(Z < 65|F)}$$

$$\mathbf{P}(X < 65|M) = \mathbf{P}(Z \leq \frac{65 - 69.1}{2.9}) = \mathbf{P}(Z \leq -1.41) = 1-\mathbf{P}(Z < 1.41) = 1-0.9207 = 0.0793$$

$$\mathbf{P}(X < 65|F) = \mathbf{P}(Z \leq \frac{65 - 63.7}{2.7}) = \mathbf{P}(Z \leq 0.48) = 0.6844$$

$$\rightarrow \mathbf{P}(M|X < 65) = \frac{0.0793}{0.0793 + 0.6844} = 0.1038$$

Wrong formula/lookup—point off. Wrong Bayes rule point off.

3. (a) Either X or $1 - X$ is the smaller segment. When $X \leq (1 - X)/4$ $X \leq 1/5$, or when $X \geq 4/5$ the smaller segments are less than $1/4$ of the longer. So the answer is $2/5$.

   (b) $P(1-X \leq t) = P(X \geq 1-t)$. When $t \in [0, 1]$ this is $1-(1-t) = t$. It is also 0 when $t \leq 0$ and 1 when $t > 1$. When you differentiate this we get $f_{1-X}(t) = 1(t \in [0, 1])$.

4. **[Exponential distribution]** (1+2+1+2) I buy a new laptop. According to the manufacturer, the lifetime, X, of the laptop is a continuous random variable with PDF $f_X(x) = 0.5e^{-0.5x}$ for all $x > 0$.

   (a) What is the probability that the laptop fails during the first year?

$$\mathbf{P}(X \leq 1) = \int_0^1 0.5e^{-0.5x} = [-e^{-0.5x}]_0^1 = 1 - e^{-0.5} = 0.393$$

   (b) What is the expected lifetime of the laptop?

$$E[X] = \int_0^\infty 0.5xe^{-0.5x}dx = [-xe^{-0.5x} - 2e^{-0.5x}]_0^\infty = -0 - 0 + - + 2 = 2\text{years}$$

($\frac{n}{e^n} \rightarrow 0$ as $n \rightarrow \infty$, since e^n grows faster than n) Two points for doing integration by parts correctly.

(c) 3 years pass, and the laptop is still going strong. What is the conditional PDF of the total lifetime, given I know the lifetime is at least 3 years?

$$f_{X|X>3}(x) = \frac{f_X}{\mathbf{P}(X>3)}$$

$\mathbf{P}(X>3) = 1 - \int_0^3 0.5e^{-0.5}dx = 1 - [-e^{-0.5x}]_0^3 = e^{-1.5}$, so

$$f_{X|X>3}(x) = \begin{cases} \frac{0.5e^{-0.5x}}{e^{-1.5}} & x \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

If we write $x = 3 + y$, this reduces to

$$f_{Y|Y>0}(y) = \begin{cases} \frac{0.5e^{-1.5-0.5y}}{e^{-1.5}} = 0.5e^{-0.5y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(this step isn't required) Point off for any mistake.

(d) Based on this conditional PDF, how many more years do I expect it to last? From above, we see the conditional PDF for the number of additional years is the same as the original PDF, so we expect it to last another 2 years. (Or, doing the integral is fine). If they do the calculation right give full points. If they do $E[X|X \geq 3]$ and do not subtract 3 from it, don't take off a point, just make a note.

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990