Practice Quiz 2

SDS 321

1. I write down the numbers 1, 3, 3, and 7 on separate pieces of paper and put them in a hat. I then draw one at random. Let X be a random variable, where X is the number I draw from the hat.

   (a) What is the pmf of X?

   $$P(X = x) = \begin{cases} 0.25 & \text{if } x = 1 \\ 0.50 & \text{if } x = 3 \\ 0.25 & \text{if } x = 7 \\ 0 & \text{otherwise} \end{cases}$$

   (b) What is $P(X < 3)$?

   $$P(X < 3x) = 0.25$$

   (c) What is the expectation of X?

   $$E[X] = 1 \cdot 0.25 + 3 \cdot 0.5 + 7 \cdot 0.25 = 3.5$$

   (d) What is the variance of X?

   First find $E[X^2]$

   $$E[X^2] = 1^2 \cdot 0.25 + 3^2 \cdot 0.5 + 7^2 \cdot 0.25 = 17$$

   Then:

   $$Var(X) = E[X^2] - E[X]^2 = 17 - 3.5^2 = 4.75$$

   (e) Let $Y = 2X$, what is the pmf of Y?

   $$P(Y = y) = \begin{cases} 0.25 & \text{if } y = 2 \\ 0.50 & \text{if } y = 6 \\ 0.25 & \text{if } y = 14 \\ 0 & \text{otherwise} \end{cases}$$

2. The owner of a chicken farm has several chickens, and has recorded statistics on how often they lay eggs.

(a) If the probability of laying a brown egg is 0.6. If a hen lays 7 eggs, what is the probability the hen lays exactly 3 brown eggs?

This can be modeled by a binomial distribution:

$$P(X = 7) = \binom{7}{3} 0.6^3 0.4^4 \approx 0.19$$

(b) The probability that a hen lays any eggs on a given day is 0.9. If the farmer starts observing a chicken every day, what is the probability that the chicken lays their first egg on the second day of observation?

This can be modeled by a geometric distribution:

$$P(X = 2) = 0.1^1 \cdot 0.9^1 = 0.09$$

(c) If the number of eggs a hen lays on a given day follows a poisson distribution with a mean of 3, what is the probability the hen lays 2 or fewer eggs that day?

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!} \approx 0.42$$

3. Consider the following joint pmf of X and Y:

	$X = 1$	$X = 2$
$Y = 0$	0.1	0.3
$Y = 1$	0.2	0.2
$Y = 2$	0.1	0.1

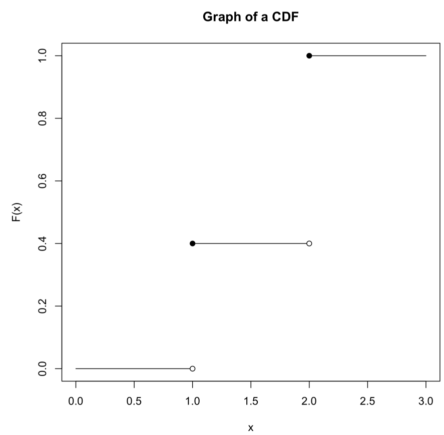
(a) What is $P(X \leq 1, Y > 1)$?

$$P(X \leq 1, Y > 1) = P(X = 1, Y = 2) = 0.1$$

(b) Find the marginal pmf of X.

$$P(X = x) = \begin{cases} 0.4 & \text{if } x = 1 \\ 0.6 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find and sketch the cdf of X.



(d) Find the conditional pmf of $Y \mid X = 1$.

$$P(Y = y \mid X = 1) = \begin{cases} 0.25 & \text{if } y = 0 \\ 0.50 & \text{if } y = 1 \\ 0.25 & \text{if } y = 2 \\ 0 & \text{otherwise} \end{cases}$$

(e) What is the expectation of $Y \mid X = 1$?

$$E[Y \mid X = 1] = 0 \cdot 0.25 + 1 \cdot 0.50 + 2 \cdot 0.25 = 1.$$

(f) What is the pdf of $Z = X - Y$?

$$P(Z = z) = \begin{cases} 0.1 & \text{if } z = -1 \\ 0.3 & \text{if } z = 0 \\ 0.3 & \text{if } z = 1 \\ 0.3 & \text{if } z = 2 \\ 0 & \text{otherwise} \end{cases}$$

4. Let $E[X] = 1$, $E[Y] = 2$, $E[X^2] = 2$ and $E[Y^2] = 4$. Assuming X and Y are independent, find the following:

   (a) $E[2Y + 3]$

   $$E[2Y + 3] = 2E[Y] + 3 = 2 \cdot 2 + 3 = 7$$

   (b) $Var(X - 1)$

   $$Var(X - 1) = Var(X) = E[X^2] - E[X]^2 = 2 - 1 = 1$$

   (c) $E[2X - Y]$

   $$E[2X - Y] = 2E[X] - E[Y] = 2 \cdot 1 - 2 = 0$$

   (d) $E[3XY]$

   $$E[3XY] = 3E[X]E[Y] = 3 \cdot 1 \cdot 2 = 6$$

   (e) $Var(2X + Y - 1)$

   $$\begin{aligned} Var(2X + Y - 1) &= Var(2X + Y) \\ &= Var(2x) + Var(Y) \\ &= 4Var(X) + Var(Y) \\ &= 4(E[X^2] - E[X]^2) + E[Y^2] - E[Y]^2 \\ &= 4(2 - 1) + 4 - 2^2 = 4 \end{aligned}$$

5. Consider the following pdf:

$$f_x(x) = \begin{cases} cx, & \text{if } 0 \leq x \leq \frac{1}{3} \\ 0, & \text{otherwise} \end{cases}$$

(a) What value of c makes this a valid probability distribution?

$$\int_0^{\frac{1}{3}} cx \; dx = 1$$
$$\frac{c}{2}x^2\Big|_0^{\frac{1}{3}} = 1$$
$$\frac{c}{18} = 1$$
$$c = 18$$

(b) What is the expectation of X?

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$
$$= \int_0^{\frac{1}{3}} 18x^2 \; dx$$
$$= 6x^3\Big|_0^{\frac{1}{3}}$$
$$= \tfrac{6}{27} = \tfrac{2}{9}$$

(c) What is $P(X < \frac{1}{9})$?

$$P(X < \tfrac{1}{9}) = \int_0^{\frac{1}{9}} 18x \; dx$$
$$= 9x^2\Big|_0^{\frac{1}{9}}$$
$$= \tfrac{1}{9}$$

6. Consider the following cdf:

$$F_x(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{3}x^2, & \text{if } 0 \le x \le \frac{3}{2} \\ 1, & \text{if } \frac{3}{2} < x \end{cases}$$

(a) What is the pdf of X?

$$f_x(x) = \begin{cases} \frac{2}{3}x, & \text{if } 0 \leq x \leq \frac{3}{2} \\ 0, & \text{otherwise} \end{cases}$$

(b) Is X continuous or discrete?

X is continuous.

7. A student reports that they get between 6 and 10 hours of sleep each night, with any time in that interval being equally likely.

(a) Write the pdf for X, the amount of sleep they get in a night. What type of distribution is this?

This is a uniform distribution:

$$f_x(x) = \begin{cases} \frac{1}{10-6} = \frac{1}{4}, & \text{if } 6 \leq x \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

(b) What is the average number of hours of sleep they get in a night?

$$E[X] = \int_6^{10} \frac{x}{4} dx = \frac{x^2}{8}\Big|_6^{10} = \frac{100-36}{8} = 8$$

(c) What is the probability they get more than 7 hours of sleep in a night?

$$P(X > 7) = \int_7^{10} \frac{1}{4} dx = \frac{x}{4}\Big|_7^{10} = \frac{3}{4} = 0.75$$

(d) After tracking their sleep, the student believes a normal distribution with a mean of 8 and a standard deviation of 1 better fits their sleep patterns. What is the probability they get more than 7 hours of sleep in a night under this new model?

$$P(X > 7) = 1 - P(X < 7) = 1 - P\left(Z < \frac{7-8}{1} = -1\right) \approx 0.84$$

(e) During the sleep tracking, the student had one night where they got less than six hours of sleep because the smoke detector went off and they had to replace the batteries. If the number of months until the smoke detector goes off due to the batteries follows an exponential distribution with a mean of 6 months, what is the probability the student will have to change the batteries again some time in the next 5 months?

$$P(X < 5) = \int_0^5 \tfrac{1}{6} e^{-\frac{x}{6}} dx = -e^{-\frac{x}{6}} \Big|_0^5 = 1 - e^{-\frac{5}{6}} \approx 0.57$$

8. Consider the following two pdfs:

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2}y, & \text{if } 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

If X and Y are independent, what is the joint pdf $f_{X,Y}(x, y)$?

$$f_{X,Y}(x, y) = \begin{cases} xy, & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

9. Consider the following joint pdf:

$$f_{X,Y}(x, y) = \begin{cases} \frac{2}{9} & \text{for } 0 \leq x \leq 3, 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal pdf of X ($f_X(x)$).

$$f_X(x) = \begin{cases} \int_0^x \frac{2}{9} dy = \frac{2x}{9} & \text{for } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) What is the expectation of X?

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_0^3 x \frac{2x}{9}\, dx \\ &= \int_0^3 \frac{2x^2}{9}\, dx \\ &= \frac{2x^3}{27}\Big|_0^3 \\ &= 2 \end{aligned}$$

(c) What is the variance of X?

First let us find $E[X^2]$:

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_0^3 \frac{2x^3}{9}\, dx \\ &= \frac{x^4}{18}\Big|_0^3 \\ &= \frac{9}{2} \end{aligned}$$

Then:

$$Var(X) = E[X^2] - E[X]^2 = \frac{9}{2} - 2^2 = 0.5$$

(d) What is the conditional distribution of $X \mid Y$ ($f_{X|Y}(x \mid y)$)?

First we need to find $f_Y(y)$

$$f_Y(y) = \begin{cases} \int_y^3 \frac{2}{9} dx = \frac{6-2y}{9} & \text{for } 0 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$f_{X|Y}(x \mid y) = \begin{cases} \frac{2}{6-2y} & \text{for } y \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(e) What is the expectation of X given $Y = 1$ ($E[X \mid Y = 1]$)?

$$E[X \mid Y = 1] = \int_1^3 x \frac{2}{6-2} dx = \int_1^3 \frac{x}{2} dx = \frac{x^2}{4}\Big|_1^3 = \frac{9-1}{4} = 2$$