THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics
Lecture 16: Continuous random variables

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Roadmap

- Two random variables: joint distributions
    - Derived distributions
        - Linear functions
        - Monotonic functions

Functions of random variables

- If X is a random variable, we know that for any function g, $Y = g(X)$ is also a random variable.
    - Each outcome in our sample space still maps to a number.
- For example let $X \sim \mathit{Binomial}(5, 0.5)$, i.e. $p_X(k) = \binom{5}{k} 0.5^5$.
- What is the PMF of $Y = X^2$?

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- $p_Y(y) = P(X^2 = y) = P(X = \sqrt{y}) = \binom{5}{\sqrt{y}} 0.5^5$
- But $(-\sqrt{y})^2 = y$ too, why didn't we take that into account?

Functions of random variables

- For continuous random variables, it's a bit harder.
- Easier to think about $P(g(X) \leq y)$... i.e. the CDF of Y.
- We can then differentiate $F_Y(y)$ to get the PDF of y.

Functions of continuous random variables: Example

- What is the PDF of the area of a circle, if the radius is a uniform random variable on [0,1]?
- If X is the radius, we know that the area $Y = \pi X^2$.
- We know that

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

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- $f_X(x)$ is only positive for $0 \leq x \leq 1$, corresponding to $0 \leq y \leq \pi$. So,

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \int_0^{\sqrt{y/\pi}} f_X(x)dx = \sqrt{\frac{y}{\pi}} & 0 \leq y \leq \pi \\ 1 & y > \pi \end{cases}$$

Functions of continuous random variables: Example

- From the previous slide:

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \int_0^{\sqrt{y/\pi}} f_X(x)dx = \sqrt{\frac{y}{\pi}} & 0 \leq y \leq \pi \\ 1 & y > \pi \end{cases}$$

- We can differentiate to get

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{d}{dy}\left(\sqrt{\frac{y}{\pi}}\right) = \frac{1}{2}\sqrt{\frac{1}{y\pi}} & 0 \leq y \leq \pi \\ 0 & y > \pi \end{cases}$$

Try it yourself: linear function of continuous random variables

- Let X be a random variable with PDF $f_X(x)$, and let $Y = 2X + 3$
- Then the CDF of Y is given by:

$$
\begin{aligned}
F_Y(y) = P(2X + 3 \leq y) = P\left(X \leq \frac{y-3}{2}\right) \\
= F_X\left(\frac{y-3}{2}\right)
\end{aligned}
$$

- Differentiating, we get:

$$
\begin{aligned}
f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X\left(\frac{y-3}{2}\right)}{dy} \\
= \frac{1}{2} f_X\left(\frac{y-3}{2}\right)
\end{aligned}
$$

Try it yourself: linear function of continuous random variables

- Let X be a random variable with PDF $f_X(x)$, and let $Y = -2X + 3$
- Then the CDF of Y is given by:

$$F_Y(y) = P(-2X + 3 \leq y) = P\left(X \geq \frac{3-y}{2}\right)$$
$$= 1 - F_X\left(\frac{3-y}{2}\right)$$

- Differentiating, we get:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -\frac{dF_X\left(\frac{3-y}{2}\right)}{dy}$$
$$= \frac{1}{2} f_X\left(\frac{y-3}{(-2)}\right)$$

Functions of continuous random variables

- Let X be a random variable with PDF $f_X(x)$, and let $Y = aX + b$
-

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a > 0 \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Linear functions of continuous random variables

- So, for any continuous random variable X, if $Y = aX + b$, then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- Let's consider a normal random variable... $X \sim N(0,1)$, then what is the distribution of $Y = aX + b$?

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- So,

$$\begin{aligned} f_Y(y) =& \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}} \exp\{-\left(\frac{y-b}{a}\right)^2/2\} \\ =& \frac{1}{\sqrt{2\pi}|a|} \exp\left(-\frac{(y-b)^2}{2a^2}\right) \end{aligned}$$

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- This is just the PDF of a normal distribution with mean b and variance a^2!
- In fact, if $X \sim N(\mu, \sigma^2)$, then $Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$.