SDS321 Practice Problems (Answers)

1. How many permutations of ALGORITHM have the A, L, and G together in any order? $3!7!$

2. A byte consists of eight bits, each bit being a 0 or 1.
   (a) How many bytes contain exactly three 1 's? $\mathrm{C}(8,3)$
   (b) How many bytes contain at least one 1 ? $2^8 - 1$
   (c) How many bytes end with three 1 's or begin with two 0 's? $2^5 + 2^6 - 2^3$
   (d) How many bytes contain at least three 1 's and at least two 0 's? $\mathrm{C}(8,3)+\mathrm{C}(8,4)+\mathrm{C}(8,5)+\mathrm{C}(8,6)$

3. For $n > 0$, consider strings of length 2n of 0 's and 1's. Assuming all such strings are equally likely, what is the probability that one such string has an equal number of 0's and 1's $\mathrm{C}(2\mathrm{n},\mathrm{n})/2^{2\mathrm{n}}$

4. How many ways are there to choose a half dozen donuts from 10 varieties
   (a) if there are no two donuts of the same variety? $\mathrm{C}(10,6)$
   (b) if there are at least two varieties? $\mathrm{C}(15,6) - 10$
   (c) if there must be at least one but no more than 4 glazed? $\mathrm{C}(14,5) - \mathrm{C}(10,1)$

5. Given a set $\mathrm{A} = \{\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d},\mathrm{e}\}$,
   (a) How many different sequences (using the elements of A) of length $\mathrm{n} > 0$ exist that contain at most one a? $4^{\mathrm{n}} + \mathrm{n}4^{\mathrm{n}-1}$
   (b) How many subsets of A are there? 2^5
   (c) How many non-empty subsets of A are there? $2^5 - 1$
   (d) How many subsets of size 3 can you create from A? $\mathrm{C}(5,3)$
   (e) How many subsets of A are there that are entirely vowels or entirely consonants? (the empty set satisfies both) $2^2 + 2^3 - 1$
   (f) How many subsets of A are there that have at least one vowel and one consonant? This is: all – the number in (e), in particular: $2^5 - \left(2^2 + 2^3 - 1\right)$
   (g) How many subsets of A of size 3 contain exactly one vowel? $\mathrm{C}(2,1)\mathrm{C}(3,2)$
   (h) How many ways can you arrange the letters in A? $5!$
   (i) How many ways can the letters of A be arranged so that all of the vowels are together? $2!4!$
   (j) How many ways can you arrange the letters of A so that it is not the case that all of the vowels are together? $5! - 2!4!$
   (k) How many ways can you arrange the letters in A so that vowels and consonants alternate and the arrangement begins with a consonant? $3 \cdot 2 \cdot 2 \cdot 1 \cdot 1$

   For $\mathrm{n} > 0$, assume all strings of length n from the set A (allowing repetition) are equally likely.
   (l) What is the probability that such a string has no a? $4^{\mathrm{n}}/5^{\mathrm{n}}$
   (m) What is the probability that such a string has no b given that it has no a? $3^{\mathrm{n}}/4^{\mathrm{n}}$

6. How many distinct permutations are there of the letters in "perfect"? $7!/2!$

7. How many distinct permutations are there of the digits in 1121231234 ? $10!/(4!3!2!1!)$

8. For $n \geq 5$, consider strings of length n using elements of $\{a,b,c,d,e\}$. Assume all such strings are equally likely.

(a) What is the probability that a string selected at random has exactly three a's? $\mathrm{C(n,3)} \cdot 4^{\mathrm{n}-3}/5^{\mathrm{n}}$

(b) What is the probability that such a string has exactly two b's given that it has exactly three a's?

$$\left(\mathrm{C(n,3)} \cdot \mathrm{C(n-3,2)} \cdot 3^{\mathrm{n}-5}\right) / \left(\mathrm{C(n,3)} \cdot 4^{\mathrm{n}-3}\right) = \mathrm{C(n-3,2)} \cdot 3^{\mathrm{n}-5}/4^{\mathrm{n}-3}$$

9. How many distinct permutations are there of the letters in "breeze"? 120

10. Three pollsters canvas 21 houses. Each pollster will visit seven houses. How many different assignments of pollsters to houses are possible? $21!/\,(7! \cdot 7! \cdot 7!)$

11. Suppose that a certain precinct contains 350 voters of which 250 are Democrat and 100 are Republican. If 30 voters are chosen at random from the precinct, what is the probability that exactly 18 Democrats will be selected?

$$\mathrm{C(250,18)} \cdot \mathrm{C(100,12)/C(350,30)}$$

12. Would it be rational for an agent to hold the following three beliefs?

$$\mathrm{P(E) = .3, P(F) = .4, \text{ and } P(E \cup F) = .8 \text{ ? Explain.}}$$

No. Since $P(E \cup F)$ must be no more than $P(E) + P(F)$. Otherwise, $P(E \cap F)$ would have to be < 0, which is not possible.

13. Show $P(E \cap F) \geq P(E) + P(F) - 1$.

Since $1 = P(S) \geq P(E \cup F) = P(E) + P(F) - P(E \cap F)$,

Therefore $\mathrm{P(E \cap F) \geq P(E) + P(F) - 1}$.

14. Let X and Y take on values 1 or -1. Let

$$\begin{aligned}
\mathrm{p(1,1)} &= \mathrm{P(X = 1, Y = 1)} \\
\mathrm{p(-1,1)} &= \mathrm{P(X = -1, Y = 1)} \\
\mathrm{p(1,-1)} &= \mathrm{P(X = 1, Y = -1)} \\
\mathrm{p(-1,-1)} &= \mathrm{P(X = -1, Y = -1)}
\end{aligned}$$

Suppose that $\mathrm{E[X]} = 0$ and $\mathrm{E[Y]} = 0$. Show that

$$\begin{aligned}
\mathrm{p(1,1)} &= \mathrm{p(-1,-1)} \\
\mathrm{p(-1,1)} &= \mathrm{p(1,-1)}
\end{aligned}$$

Let $p = p(1,1)$. Find:

(a) Var[X]

(b) Var[Y]

(c) Cov[X, Y] where $\mathrm{Cov[X,Y] = E[XY] - E[X]E[Y]}$

First:

$$\begin{aligned}
&p_X(1) = p(1,1) + p(1,-1)\\
&p_X(-1) = p(-1,1) + p(-1,-1)\\
&\text{So } E[X] = p(1,1) + p(1,-1) - p(-1,1) - p(-1,-1) \text{ and}\\
&p_Y(1) = p(1,1) + p(-1,1)\\
&p_Y(-1) = p(1,-1) + p(-1,-1)
\end{aligned}$$

So $E[Y] = p(1,1) - p(1,-1) + p(-1,1) - \mathrm{p}(-1,-1)$.

Notice that $\mathrm{E[X]} + \mathrm{E[Y]} = 2\mathrm{p}(1,1) - 2\mathrm{p}(-1,-1) = 0$ so $\mathrm{p}(1,1) = \mathrm{p}(-1,-1)$.

And $\mathrm{E[X]} - \mathrm{E[Y]} = 2\mathrm{p}(1,-1) - 2\mathrm{p}(-1,1) = 0$ so $\mathrm{p}(-1,1) = \mathrm{p}(1,-1)$.

Let $\mathrm{p} = \mathrm{p}(1,1)$. Find

(a)

$$\begin{aligned}
\mathrm{Var[X]} &= \mathrm{E}\left[\mathrm{X}^2\right] - \mathrm{E}^2[\mathrm{X}] = \mathrm{E}\left[\mathrm{X}^2\right] - 0\\
&= \mathrm{p}(1,1) + \mathrm{p}(1,-1) + \mathrm{p}(-1,1) + \mathrm{p}(-1,-1) = 1
\end{aligned}$$

(b) Similarly, $\mathrm{Var[Y]} = 1$

(c)

$$\begin{aligned}
\mathrm{Cov[X,Y]} &= \mathrm{E[XY]} - \mathrm{E[X]E[Y]} = \mathrm{E[XY]} - 0\\
&= 1\cdot\mathrm{p}(1,1) + (-1)\cdot\mathrm{p}(1,-1) + (-1)\cdot\mathrm{p}(-1,1) + (1)\cdot\mathrm{p}(-1,-1)\\
&= \mathrm{p} - (1/2 - \mathrm{p}) - (1/2 - \mathrm{p}) + \mathrm{p} = 4\mathrm{p} - 1
\end{aligned}$$

15. Given $f(x,y) = 3x$ for $0 \le y \le x \le 1$, find

(a) E(X)

$$f(x) = \int_0^x 3x\,dy = 3xy|_0^x = 3x^2 \quad \text{for } 0 \le \mathrm{x} \le 1$$

$$0 \text{ otherwise}$$

$$E[X] = \int_0^1 x3x^2dx = \left.\frac{3x^4}{4}\right|_0^1 = \frac{3}{4}$$

(b) E(Y)

$$f(y) = \int_y^1 3x\,dx = \left.\frac{3x^2}{2}\right|_y^1 = \frac{3}{2}\left(1 - y^2\right) \quad \text{for } 0 \le \mathrm{y} \le 1$$

$$0 \text{ otherwise}$$

$$E[Y] = \int_0^1 y\frac{3}{2}\left(1 - y^2\right)dy = \left.\left(\frac{3}{4}y^2 - \frac{3}{8}y^4\right)\right|_0^1 = \frac{3}{8}$$

(c) Var(X)

$$E\left[X^2\right] = \int_0^1 x^2 3x^2 dx = \left.\frac{3x^5}{5}\right|_0^1 = \frac{3}{5}$$

So $\mathrm{Var(X)} = 3/5 - (3/4)^2 = (48 - 45)/80 = 3/80$.

(d) Cov(X, Y)

$$E[XY] = \int_0^1\int_0^x 3x^2y\,dy\,dx = \int_0^1 \frac{3}{2}x^4dx = \frac{3}{10}$$

So $\mathrm{Cov}(X,Y) = (3/10) - (3/4)(3/8) = (3/10) - (9/32) = (48 - 45)/160 = 3/160$.

16. Assume that any result produced by a certain routine will be faulty with probability 0.1, independently. What is the probability that a sample of three will have at most 1 faulty result?

$$\mathrm{P}(\text{ none or exactly one }) = .9^3 + 3(.1)(.9)^2$$

17. Suppose the number of bugs in a routine has a Poisson Distribution with parameter $\lambda = 1$. What's the probability that a routine has at least one bug?

    Let X be the number of bugs

$$\mathrm{P}(\mathrm{X} \geq 1) = 1 - \mathrm{P}(\mathrm{X} = 0) = 1 - 1^0/0\,!^{*\ -1} = 1 - (1/\mathrm{e})$$

18. What is the distribution of the sum of two i.i.d. Poisson Distributions? Let $X \sim \mathrm{Pois}(\lambda)$ and $\mathrm{Y} \sim \mathrm{Pois}(\lambda)$. Let $\mathrm{Z} = \mathrm{X} + \mathrm{Y}$. Then $\mathrm{P}(\mathrm{Z} = \mathrm{z}) = \mathrm{P}(\mathrm{X} + \mathrm{Y} = \mathrm{z}) = ?$

$$\sum_{x=0}^{z} p_{X,Y}(x, z-x) = \sum_{x=0}^{z} \frac{\lambda^x}{x!} e^{-\lambda} \frac{\lambda^{z-x}}{(z-x)!} e^{-\lambda}$$
$$= \sum_{x=0}^{z} \frac{\lambda^z}{x!(z-x)!} e^{-2\lambda} \qquad \text{(I factored values from sum)}$$
$$= \lambda^z e^{-2\lambda} \sum_{x=0}^{z} \frac{1}{x!(z-x)!} \qquad \text{(I try to put this in the form to use the Binomial theorem)}$$
$$= \frac{\lambda^z e^{-2\lambda}}{z!} \sum_{x=0}^{z} \left(\frac{z!}{x!(z-x)!} 1^x 1^{z-x} \right) \qquad \left(\text{Binomial Theorem } \sum_{x=1}^{z} \binom{z}{x} 1^x 1^{z-x} = (1+1)^z \right)$$
$$= \frac{\lambda^z e^{-2\lambda}}{z!} 2^z$$
$$= \frac{(2\lambda)^z e^{-2\lambda}}{z!}$$

    So $\mathrm{Z} \sim \mathrm{Pois}(2\lambda)$.

19. A bag contains n pairs of shoes; each pair is a different style. You pick three random shoes from the bag.

    (a) What is the probability of having a pair of shoes among the three you picked? $1 - \mathrm{P}(\text{no pair}) = 1 - (2\mathrm{n} \cdot 2(\mathrm{n}-1) \cdot 2(\mathrm{n}-2)/(2\mathrm{n} \cdot (2\mathrm{n}-1) \cdot (2\mathrm{n}-2)))$

    (b) What is the probability there is at least one left and at least one right shoe among the three? $1 - \mathrm{P}(\text{all left or all right}) = 1 - 2 \cdot \mathrm{C}(\mathrm{n}, 3)/\mathrm{C}(2\mathrm{n}, 3)$

20. Consider a novel four-sided die with faces numbered 1, 2, 3, and 4. The PMF for any one roll of this die is

$$\mathrm{p_X(x)} = \begin{cases} 1/2 \text{ for } \mathrm{x} = 1 \\ 1/6 \text{ for } \mathrm{x} = 2, 3, 4 \end{cases}$$

    Consider a sequence of six independent rolls of this die, and let Xi be the random variable corresponding to the i-th roll.

    (a) What is the probability that exactly three of the rolls are equal to 3 ? $\mathrm{C}(6,3)(1/6)^3(5/6)^3$

    (b) What is the probability that the first roll is 1, given that exactly two of the six rolls are equal to 1? $\left(1/2 \cdot \mathrm{C}(5,1) \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^4\right) / \left(\mathrm{C}(6,2)(1/2)^2 \cdot (1/2)^4\right) = 1/3$

21. Consider 8 independent tosses of a 4-sided fair die. Let X_i be the number of tosses that result in i, $\mathrm{i} = 1, 2, 3, 4$.

(a) Are X_1 and X_2 uncorrelated, positively correlated, or negatively correlated? Give a one-line "intuitive" justification.

Negatively since if one is high (goes up) the other is likely to be low (go down).

(b) Compute the covariance of X_1 and X_2 : cov (X_1, X_2).

$$\mathrm{P}\left(\mathrm{X}_1 = \mathrm{x}\right) = \mathrm{C}(8, \mathrm{x})(1/4)^{\mathrm{x}}(3/4)^{8-\mathrm{x}}$$
$$\mathrm{P}\left(\mathrm{X}_2 = \mathrm{y}\right) = \mathrm{C}(8, \mathrm{y})(1/4)^{\mathrm{y}}(3/4)^{8-\mathrm{y}}$$
$$\mathrm{E}\left[\mathrm{X}_1\right] = \mathrm{E}\left[\mathrm{X}_2\right] = 2$$
$$\mathrm{P}\left(\mathrm{X}_1 = \mathrm{x} \text{ and } \mathrm{X}_2 = \mathrm{y}\right) = \mathrm{C}(8, \mathrm{x})\mathrm{C}(8-\mathrm{x}, \mathrm{y})(1/4)^{\mathrm{x}+\mathrm{y}}(1/2)^{8-\mathrm{x}-\mathrm{y}}$$
$$E\left[X_1 X_2\right] = \sum_{x=0}^{8}\sum_{y=0}^{8-x} xy \frac{(8-x)!}{y!(8-x-y)!}\frac{8!}{x!(8-x)!}\left(\frac{1}{4}\right)^x\left(\frac{1}{4}\right)^y\left(\frac{1}{2}\right)^{8-x-y}$$
$$= \sum_{x=0}^{8} x\left(\frac{1}{4}\right)^x \frac{8!}{x!(8-x)!}\sum_{y=0}^{8-x} y\frac{(8-x)!}{y!(8-x-y)!}\left(\frac{1}{4}\right)^y\left(\frac{1}{2}\right)^{8-x-y}$$
$$= \sum_{x=0}^{8} x\left(\frac{1}{4}\right)^x \frac{8!}{x!(8-x)!}\frac{8-x}{4}\sum_{y=1}^{8-x}\frac{(8-x-1)!}{(y-1)!(8-x-y)!}\left(\frac{1}{4}\right)^{y-1}\left(\frac{1}{2}\right)^{8-x-y}$$
$$= \sum_{x=0}^{8} x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{8-x-1}\frac{8!}{x!(8-x)!}\frac{8-x}{4}$$
$$= \sum_{x=1}^{7}\frac{8!}{(x-1)!(7-x)!}\frac{1}{4}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{8-x-1}$$
$$= \frac{8\cdot 7}{4^2}\sum_{x=1}^{7}\frac{6!}{(x-1)!(7-x)!}\left(\frac{1}{4}\right)^{x-1}\left(\frac{3}{4}\right)^{7-x}$$
$$= \frac{8\cdot 7}{4^2} = \frac{7}{2}$$

Hence the cov $\left(\mathrm{X}_1, \mathrm{X}_2\right) = 7/2 - 2\cdot 2 = -1/2$

22. Consider results from the toss of 100 fair coins and let H count the number of heads.

(a) Use Markov's Inequality to estimate the probability of H being at least 60 . $\mathrm{P}(\mathrm{H} \geq 60) \leq 50/60 = 5/6$

(b) Use Chebyshev's inequality to estimate the probability that $40 < \mathrm{H} < 60$. $\mathrm{P}(40 \leq \mathrm{H} \leq 60) \geq 1 - 25/121 = 96/121$

(c) Use the Central limit Theorem to estimate the probability that $40 < \mathrm{H} < 60$. $\mathrm{P}(40 \leq \mathrm{H} \leq 60) = \mathrm{P}(-2 \leq \mathrm{Z} \leq 2)$ approximately .95

(d) Compare results of parts a, b, and c. Markov weakest but requires least knowledge, Chebyshev next, CLT best approximation

23. Let X and Y be random variables such that $E[X] = 2, E[Y] = 5, E\left[X^2\right] = 5, E\left[Y^2\right] = 29$, $\mathrm{E[XY]} = 11$. Let $\mathrm{Z} = 2\mathrm{X} - \mathrm{Y}$. Find E[Z] and Var[Z].

$\mathrm{Var[X]} = 1, \mathrm{Var[Y]} = 4, \mathrm{Cov[X, Y]} = 1$

$\mathrm{E[Z]} = 4 - 5 = -1$ and $\mathrm{Var[Z]} = 4\,\mathrm{Var[X]} + \mathrm{Var[Y]} - 2\cdot 2\cdot \mathrm{Cov[X, Y]} = 4$

24. A pulse of light has energy X that is a random variable with parameter λ, with its PDF given as

$$f_X(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

This pulse illuminates an ideal photon-counting detector whose output N when $X = x$ is given by the conditional PMF:

$$p_{N|X}(n \mid x) = \begin{cases} \frac{x^n e^{-x}}{n!} & \text{for } n = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Useful integral and facts:

$$\int_0^\infty y^k e^{-\alpha y} dy = \frac{k!}{\alpha^{k+1}}, \text{ for } \alpha > 0, \text{ and } k = 0, 1, 2, \ldots \text{(recall that } 0! = 1)$$

(a) Find $E[X]$, and $\mathrm{Var}(X)$, the unconditional mean and variance of X.

(b) Find $E[N]$ and $\mathrm{Var}[N]$, the unconditional mean and variance of N.

$$\begin{aligned} E[X] &= \int_0^\infty x \times \lambda^2 x e^{-\lambda x} dx \\ &= \int_0^\infty x^2 \lambda^2 e^{-\lambda x} dx \\ &= \frac{\lambda^2 2!}{\lambda^3} \\ &= \frac{2}{\lambda} \end{aligned}$$

$$\begin{aligned} E\left[X^2\right] &= \int_0^\infty x^2 \times \lambda^2 x e^{-\lambda x} dx \\ &= \int_0^\infty x^3 \lambda^2 e^{-\lambda x} dx \\ &= \frac{\lambda^2 3!}{\lambda^4} \\ &= \frac{6}{\lambda^2} \end{aligned}$$

So Var(X) = $2/\lambda^2$

To find the $E[N]$, we need to find $f_N(n)$. But then first we need $f_{X,N}(x, n) = f_X(x) f_{N|X}(n)$.

$$f_{X,N}(x,n) = \lambda^2 x e^{-\lambda x} \frac{x^n e^{-x}}{n!}$$

$$f_N(n) = \int_0^\infty x^{n+1} \lambda^2 \frac{e^{-(\lambda+1)x}}{n!} dx$$

$$= \frac{\lambda^2 (n+1)!}{(\lambda+1)^{n+2} n!} = \left(\begin{array}{c} n+1 \\ n \end{array} \right) \left(\left(\frac{\lambda}{\lambda+1} \right)^2 \left(\frac{1}{\lambda+1} \right)^n \right)$$

$$E(N) = \left(\frac{\lambda}{\lambda+1} \right)^2 \sum_{n=1}^{\infty} (n+1) n \left(\frac{1}{\lambda+1} \right)^n$$

$$= \left(\frac{\lambda}{\lambda+1} \right) \left(\sum_{n=1}^{\infty} n^2 \left(\frac{\lambda}{\lambda+1} \right) \left(\frac{1}{\lambda+1} \right)^n + \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\lambda+1} \right) \left(\frac{1}{\lambda+1} \right)^n \right)$$

$$= \left(\frac{\lambda}{\lambda+1} \right) \left(\frac{\frac{1}{\lambda+1}}{\left(\frac{\lambda}{\lambda+1} \right)^2} + \frac{\left(\frac{1}{\lambda+1} \right)^2}{\left(\frac{\lambda}{\lambda+1} \right)^2} + \frac{\frac{1}{\lambda+1}}{\frac{\lambda}{\lambda+1}} \right)$$

$$= \left(\frac{\lambda}{\lambda+1} \right) \left(\frac{2\lambda+2}{\lambda^2} \right)^2$$

$$= \frac{2}{\lambda}$$