THE UNIVERSITY OF TEXAS AT AUSTIN
Department of Statistics and Data Sciences
College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 9: Discrete random variables

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The Geometric random variable

- The Bernoulli PMF describes the probability of success/failure in a single trial.
- The Binomial PMF describes the probability of k successes out of n trials.
- Sometimes we may also be interested in doing trials until we see a success.

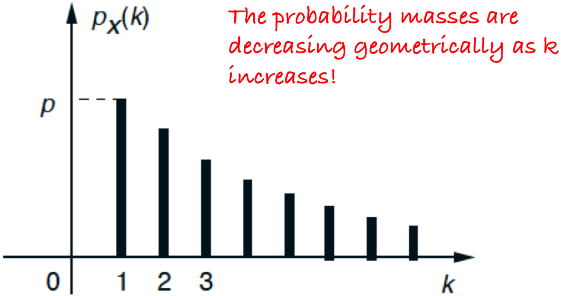
- Alice resolves to keep buying lottery tickets until he wins a hundred million dollars. She is interested in the random variable "number of lottery tickets bought until he wins the 100M$ lottery".
- Annie is trying to catch a taxi. How many occupied taxis will drive pass before she finds one that is taking passengers?

- The number of trials required to get a single success is a **Geometric Random Variable**

The geometric random variable

We repeatedly toss a biased coin ($P(\{H\}) = p$). The geometric random variable is the number X of tosses to get a head.

- X can take any integral value.
- $P(X = k) = P(\{\underbrace{TT \ldots T}_{k-1} H\}) = (1-p)^{k-1} p$.
- $\sum_k P(X = k) = 1$ (why?)



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- $P(X \geq k) = \sum_{i=k}^{\infty} p(1-p)^{i-1} = (1-p)^{k-1}$
- Intuitively, this is asking for the probability that the first $k-1$ tosses are tails.
- This probability is $P(X \geq k) = (1-p)^{k-1}$

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- Intuitively, this is asking for the probability that the first $k-1$ tosses are tails.
- This probability is $P(X \geq k) = (1-p)^{k-1}$
- $X > k$ is the event that $X \geq k+1$, and so $P(X > k) = (1-p)^{k}$

The memoryless property

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$$P(X = a + b | X > a) = \frac{P(X = a + b)}{P(X > a)}$$

- $$= \frac{p(1-p)^{a+b-1}}{(1-p)^a}$$
$$= p(1-p)^{b-1} = P(X = b)$$
- You forgot about $X > a$ and started the clock afresh!

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- $$= \frac{(1-p)^{a+b}}{(1-p)^a} = (1-p)^b$$
  $$= P(X > b)$$
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- $$\begin{aligned} P(X \leq a+b | X > a) &= \frac{P(a < X \leq a+b)}{P(X > a)} \\ &= \frac{P(X > a) - P(X > a+b)}{(1-p)^a} \\ &= \frac{(1-p)^a - (1-p)^{a+b}}{(1-p)^a} \\ &= 1 - (1-p)^b = P(X \leq b) \end{aligned}$$
- You forgot about $X > a$ and started the clock afresh!

The Poisson random variable

I have a book with 10000 words. Probability that a word has a typo is 1/1000. I am interested in how many misprints can be there on average? So a Poisson often shows up when you have a Binomial random variable with very large n and very small p but $n \times p$ is moderate. Here $np = 10$.

Our random variable might be:

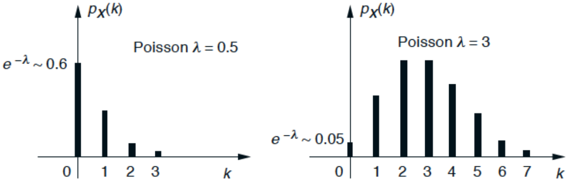
- The number of car crashes in a given day.
- The number of buses arriving within a given time period.
- The number of mutations on a strand of DNA.

We can describe such situations using a **Poisson random variable**.

The Poisson random variable

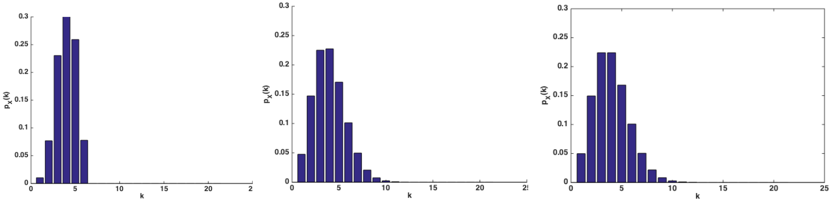
- A Poisson random variable takes non-negative integers as values. It has a nonnegative parameter λ.
- $P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$, for $k = 0, 1, 2 \ldots$.
- $\sum_{k=0}^{\infty} P(X = k) = e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots) = 1$. (Exponential series!)

The PMF is monotonically decreasing for λ=0.5



The PMF is increasing and then decreasing for λ=3

Poisson random variable



Binomial(5,0.6) Binomial(100,0.03) Poisson(3)

- When n is very large and p is very small, a binomial random variable can be well approximated by a Poisson with $\lambda = np$.
- In the above figure we increased n and decreased p so that $np = 3$.
- See how close the PMF's of the Binomial(100,0.03) and Poisson(3) are!
- More formally, we see that $\binom{n}{k} p^k (1-p)^{n-k} \approx \frac{e^{-\lambda}\lambda^k}{k!}$ when n is large, k is fixed, and p is small and $\lambda = np$.

Example

Assume that on a given day 1000 cars are out in Austin. On average, three out of 1000 cars run into a traffic accident per day.

1. What is the probability that we see at least two accidents in a day?

4. If you know there is at least one accident, what is the probability that the total number of accidents is at least two?

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1. What is the probability that we see at least two accidents in a day?
2. Use poisson approximation!
3. $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-3}(1 + 3) = 0.8$
4. If you know there is at least one accident, what is the probability that the total number of accidents is at least two?

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4. If you know there is at least one accident, what is the probability that the total number of accidents is at least two?
5. $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-3} = 0.950$.
   $P(X \geq 2 | X \geq 1) = P(X \geq 2)/P(X \geq 1) = 0.8/0.950 = 0.84$

Mean

You want to calculate average grade points from hw1. You know that 20 students got 30/30, 30 students got 25/30, and 50 students got 20/30. Whats the average?

- The average grade point is

$$\frac{30 \times 20 + 25 \times 30 + 20 \times 50}{100} = 30 \times 0.2 + 25 \times 0.3 + 20 \times 0.5$$

- Let X be a random variable which represents grade points of hw1.
- How will you calculate $P(X = 30)$?
  - See how many out of 100 students got 30 out of 30 points.
  - $P(X = 30) \approx 0.2$
  - $P(X = 25) \approx 0.3$
  - $P(X = 20) \approx 0.5$
- So roughly speaking,
  average grade $\approx 30 \times P(X = 30) + 25 \times P(X = 25) + 20 \times P(X = 20)$

Expectation

We define the expected value (or expectation or mean) of a discrete random variable X by

$$E[X] = \sum_x xP(X = x).$$

- X is a Bernoulli random variable with the following PMF:

$$P(X = x) = \begin{cases} p & X = 1 \\ 1 - p & X = 0 \end{cases}$$

So $E[X] =$.

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  So $E[X] = 1 \times p + 0 \times (1 - p) = p$.
- Expectation of a Bernoulli random variable is just the probability that it is one.
- You will also see notation like μ_X.

Expectation: example

You are tossing 4 fair coins independently. Let X denote the number of heads. What is $E[X]$?

- Any guesses? Well, on an average we should see about 2 coin tosses. No?
- Lets write down the PMF first.

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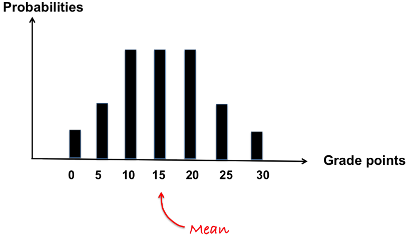
- So $E[X]=\frac{4}{2^4}+2\frac{6}{2^4}+3\frac{4}{2^4}+4\frac{1}{2^4}=\frac{32}{16}=2.$

Expectation of a function of a random variable

Lets say you want to compute $E[g(X)]$. Example, I know average temperature in Fahrenheit, but I now want it in Celsius.

- $E[g(X)] = \sum_x g(x)P(X = x)$.
- Follows from the definition of PMF of functions of random variables.
- Look at page 15 of Bersekas-Tsitsiklis and derive it at home!
- So $E[X^2] = \sum_x x^2 P(X = x)$. **Second moment of X**
- So $E[X^3] = \sum_x x^3 P(X = x)$. **Third moment of X**
- So $E[X^k] = \sum_x x^k P(X = x)$. k^{th} **moment of X**
- We are assuming "under the rugs" that all these expectations are well defined.

Expectation



- Think of expectation as center of gravity of the PMF or a representative value of X.
- How about the spread of the distribution? Is there a number for it?

Variance

Often, you may want to know the spread or variation of the grade points for homework1.

- If everyone got the same grade point, then variation is?
- If there is high variation, then we know that many students got grade points very different from the average grade point in class.
- Formally we measure this using variance of a random variable X.
- $\mathrm{var}(X) = E[(X - E[X])^2]$
- The standard deviation of X is given by $\sigma_X = \sqrt{\mathrm{var} X}$.
- Its easier to think about σ_X, since its on the same scale.
- The grade points have average 20 out of 30 with a standard deviation of 5 grade points. Roughly this means, most of the students have grade points within $[20 - 5, 20 + 5]$.

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Computing the variance

- $\mathrm{var}(X) = E[(X - E[X])^2] = \sum_x (x - E[X])^2 \, P(X = x)$
- Always remember! $E[X]$ or $E[g(X)]$ **do not depend on any particular value of** x. You can treat it as a constant. It only depends on the PMF of X.

- This can actually be made simpler.
- $\mathrm{var}(X) = E[X]^2 - (E[X])^2$.

- So you can calculate $E[X^2]$ (second moment) and then subtract the square of $E[X]$ to get the variance!

A tiny bit of algebra

$$\mathrm{var}(X) =$$

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$$= \sum_x x^2 P(X = x) + (E[X])^2 - 2(E[X])^2 = E[X^2] - (E[X])^2$$

Some simple rules– Expectation

Say you are looking at a linear function (or transformation) of your random variable X.

- $Y = aX + b$. Remember celsius to fahrenheit conversions? They are linear too!
- $E[Y] = E[aX + b] = aE[X] + b$, as simple as that! why?
  - $$\begin{aligned} E[aX + b] &= \sum_x (ax + b)P(X = x) \\ &= a\sum_x xP(X = x) + b\sum_x P(X = x) \end{aligned}$$

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How about $E[Y]$ for $Y = aX^2 + bX + c$?

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  - $E[aX^2 + bX + c] = \sum_x (ax^2 + bx + c)P(X = x)$

    $= a\sum_x x^2 P(X = x) + b\sum_x xP(X = x) + c\sum_x P(X = x)$

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- $Y = aX^3 + bX^2 + cX + d$. Can you guess what $E[Y]$ is?

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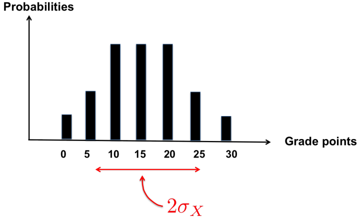
- $Y = aX^3 + bX^2 + cX + d$. Can you guess what $E[Y]$ is?

  $E[Y] = aE[X^3] + bE[X^2] + cE[X] + d$.

Some simple rules– Variance

Let $Y = X + b$. What is $\text{var}(Y)$?



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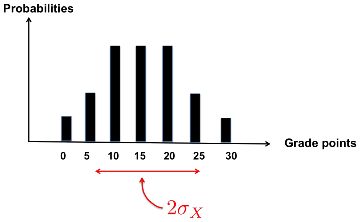

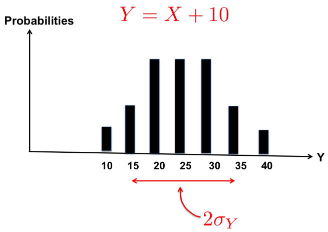

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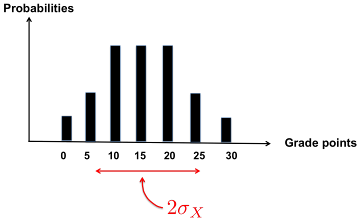

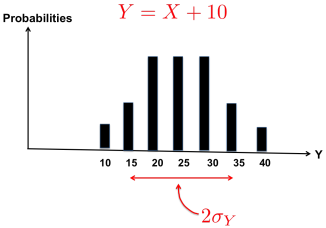

- Intuitively? Well you are just shifting everything by the same number.
- So? the spread of the numbers should stay the same!
- Prove it at home.

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Let $Y = X + b$. What is $\mathrm{var}(Y)$?

- Proof:
- $\mathrm{var}(X + b) = E\left[(X + b)^2\right] - (E[X + b])^2$
  $= E\left[X^2 + 2bX + b^2\right] - (E[X] + b)^2$

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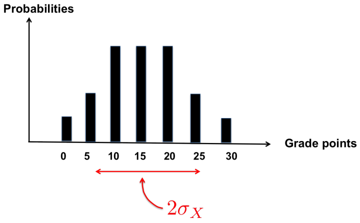

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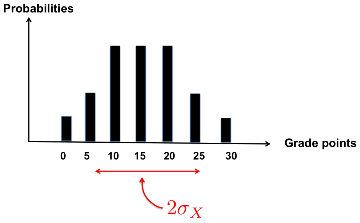

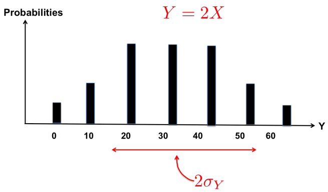

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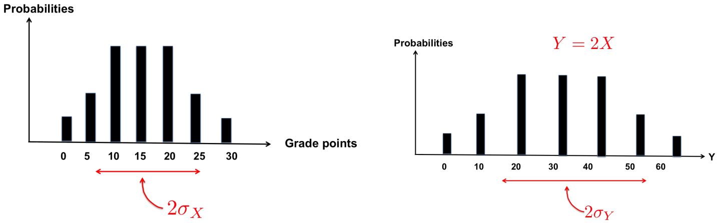

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- In general we can show that $\mathrm{var}(aX + b) = a^2\mathrm{var}(X)$.

Mean and Variance of Bernoulli

X is a Bernoulli random variable wit $P(X = 1) = p$. We saw that $E[X] = p$. What is $\mathrm{var}(X)$?

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$$E[X^2] = (1^2 \times P(X = 1) + 0^2 \times P(X = 0)) = p$$

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- $\mathrm{var}(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1 - p)$.

Mean and Variance of a Binomial

Let $X \sim Bin(n, p)$.

- $E[X] = np$ and $\mathrm{var}(X) = np(1 - p)$.
- We will derive these in the next class.

Mean and Variance of a Poisson

X has a Poisson(λ) distribution. What is its mean and variance?

- One can use algebra to show that $E[X] = \lambda$ and also $\mathrm{var}(X) = \lambda$.
- How do you remember this?
- Hint: mean and variance of the Binomial approach that of a Poisson when n is large and p is small, such that $np \approx \lambda$? Anything yet?

Mean and variance of a geometric

- The PMF of a geometric distribution is $P(X = k) = (1 - p)^{k-1} p$.
  - $E[X] = 1/p$
  - $\mathrm{var}(X) = (1 - p)/p^2$
  - We will also prove this later.