



THE UNIVERSITY OF TEXAS AT AUSTIN

Department of Statistics and Data Sciences

College of Natural Sciences

SDS 321: Introduction to Probability and Statistics

Lecture 20: Practice problems

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Continuous r.v

A continuous random variable has PDF

$$f_X(x) = \begin{cases} a + bx^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- ▶ If $E[X] = \frac{3}{5}$, find a and b .

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 - ▶ Solving the two equations we get: $a = 3/5, b = 6/5$.

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A continuous r.v. X has density function

$$f(x) = c e^{-|x|} = \begin{cases} c e^{-x} & x \geq 0 \\ c e^x & x < 0 \end{cases}$$

1. Find c .

► We have

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 c e^x dx + \int_0^{\infty} c e^{-x} dx =$$

2. Find $p(|X| > 2)$.

3. Find $E(X)$ (*Hint*: no integral required).

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► $E[X] = 0$. Why?

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▶ $E[X] = 0$. Why?

▶ **For any r.v. X with a PDF symmetric around 0, $E[X] = 0$**

Useful tricks to go around integration by parts

- ▶ You have to calculate $\int_0^{\infty} ve^{-v} dv$, but you don't remember integration by parts.
- ▶ Remember that the expectation of $X \sim \text{Exp}(\lambda)$ is

$$E[X] = \lambda \int_0^{\infty} xe^{-\lambda x} dx = \frac{1}{\lambda}$$

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- ▶ Can you get $\int_0^{\infty} 3ve^{-2v} dv$?

- ▶ $\int_0^{\infty} 3ve^{-2v} dv = 3 \int_0^{\infty} ve^{-2v} dv = 1.5 \times \int_0^{\infty} (2v)e^{-2v} dv = 1.5E[Y] = .75$
where $Y \sim \text{Exp}(2)$.

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▶ You have to calculate $\int_0^{\infty} v^2 e^{-v/2} dv$.

▶ Remember that the variance of $X \sim \text{Exp}(\lambda)$ is

$$E[X^2] = \lambda \int_0^{\infty} x^2 \exp(-\lambda x) dx = \text{var}(X) + 1/\lambda^2 = \frac{2}{\lambda^2}$$

▶ So $\int_0^{\infty} v^2 e^{-v/2} dv = 2 \int_0^{\infty} 1/2 v^2 e^{-v/2} dv$ is just $2E[X^2]$ where $X \sim \text{Exp}(1/2)$. So it equals 16.

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- ▶ You have to calculate $\int_0^\infty e^{-v^2/2} dv$.
- ▶ Whats the distribution that you should compare with?

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- ▶ You have to calculate $\int_0^{\infty} e^{-v^2/2} dv$.
- ▶ What's the distribution that you should compare with?
- ▶ $\int_0^{\infty} e^{-x^2/2} dx = P(X \geq 0) = 1/2$, where $X \sim N(0, 1)$.
- ▶ $\int_0^{\infty} e^{-v^2/2} dv = \sqrt{2\pi} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-v^2/2} dv = \sqrt{2\pi} P(X \geq 0) = \sqrt{\pi/2}$

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You have $X \sim N(0, 1)$.

▶ What is $E[X|X \geq 0]$?

▶ First, what is $f_{X|X \geq 0}(x) = f_X(x)/P(X \geq 0)$

▶ But what is $P(X \geq 0)$?

▶ So $E[X|X \geq 0] = \int_0^{\infty} x f_{X|X \geq 0}(x) dx = 2 \int_0^{\infty} x f_X(x) dx =$
 $\frac{2}{\sqrt{2\pi}} \int_0^{\infty} x e^{-x^2/2} dx$. You could integrate this.

▶ Let $v = x^2/2$. Now $E[X|X \geq 0] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-v} dv = \sqrt{2/\pi}$

▶ What is $E[X|X < 0]$

▶ You could use the same calculation. OR....

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▶ $E[X] = E[X|X \geq 0]P(X \geq 0) + E[X|X < 0]P(X < 0)$. But $E[X] = ?$

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 - ▶ Use total expectation theorem!
 - ▶ $E[X] = E[X|X \geq 0]P(X \geq 0) + E[X|X < 0]P(X < 0)$. But $E[X] = ?0$
 - ▶ $E[X|X \geq 0] \times 1/2 + E[X|X < 0] \times 1/2 = 0$. So,
 $E[X|X < 0] = -E[X|X \geq 0] = -\sqrt{2/\pi}$

Uniform r.v's

A man and a woman agree to meet at a certain location at about 12:30 p.m. Suppose the man arrives at a time uniformly distributed between 12:00 and 12:45, and the woman independently arrives at a time uniformly distributed between 12:15 and 1:00. Let X be the man's arrival time and Y be the woman's arrival time.

- ▶ What is the joint PDF?
- ▶ What is the probability that the man arrives first?

Uniform r.v's

Find the PDF first.

$$f_X(x) = \begin{cases} 1/45 & x \in [12 : 00, 12 : 45] \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1/45 & y \in [12 : 15, 1 : 00] \\ 0 & \text{otherwise} \end{cases}$$

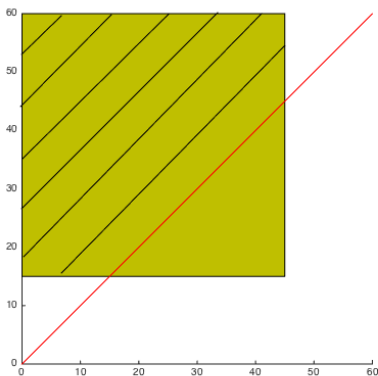
Using independence,

$$f_{X,Y}(x,y) = \begin{cases} 1/45^2 & x \in [12 : 00, 12 : 45], y \in [12 : 15, 1] \\ 0 & \text{otherwise} \end{cases}$$

Uniform r.v's

Find the probability that the man arrives first.

$$P(X < Y) = 1 - 1/45^2 \times .5 \times 30^2 = 1 - \frac{2}{9} = 7/9$$



Conditional expectation

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = 1/2$. That is, letting X denote the repair time, $X \sim \text{exponential}(\lambda)$.

- ▶ $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- ▶ $F_X(s) = P(X \leq s) = \lambda \int_0^s e^{-\lambda x} dx = 1 - e^{-\lambda s}$
- ▶ $P(X > 4) = 1 - F_X(4) = e^{-2} = .1353$

If the repair time exceeds 4 hours what is the probability that it exceeds 8 hours?

- ▶ $P(X > 8 | X > 4) =$

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- ▶ Remember! $E[X] = E[X|X \leq 4]P(X \leq 4) + E[X|X > 4]P(X > 4)$. So $2 = 1/\lambda = E[X|X \leq 4](1 - e^{-4}) + (4 + 1/\lambda)e^{-4}$. So

$$E[X|X \leq 4] = \frac{2 - 6e^{-4}}{1 - e^{-4}}$$

Correlation: Example of $|\rho| = 1$

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- ▶ Remember $X = n - Y$, so they have a linear relationship.

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$$\rho_{X, aX+b} = \frac{\text{cov}(X, aX)}{\sqrt{\text{var}(X)a^2\text{var}(X)}} = \frac{a\text{var}(X)}{|a|\text{var}(X)} = \frac{a}{|a|}$$

$$= \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}$$

More conditional expectation

$$f_X(x) = \begin{cases} 1 & x \in [-1/2, 0] \\ e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- ▶ Compute $E[X|X > 0]$.
- ▶ Compute $E[X|X \leq 0]$.
- ▶ Compute $E[X]$.

More conditional expectation

$$f_X(x) = \begin{cases} 1 & x \in [-1/2, 0] \\ e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- ▶ What is $f_{X|X>0}(x)$?

$$f_{X|X>0}(x) = \begin{cases} e^{-2x}/P(X > 0) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- ▶ $P(X > 0) = 1 - P(X \leq 0) = 1/2$.

$$f_{X|X>0}(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- ▶ So conditioned on $X > 0$, $X \sim \text{Exponential}(2)$. So $E[X|X > 0] = 1/2$.

More conditional expectation

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- ▶ What is $f_{X|X \leq 0}(x)$?

$$f_{X|X \leq 0}(x) = \begin{cases} 1/P(X < 0) = 2 & x \in [-1/2, 0] \\ 0 & \text{otherwise} \end{cases}$$

- ▶ So conditioned on $X \leq 0$, $X \sim \text{Uniform}([-1/2, 0])$. So $E[X|X < 0] = (-1/2 + 0)/2 = -1/4$.

Total expectation theorem

$$f_X(x) = \begin{cases} 1 & x \in [-1/2, 0] \\ e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

- ▶ What is $E[X]$?

$$\begin{aligned} E[X] &= E[X|X > 0]P(X > 0) + E[X|X \leq 0]P(X \leq 0) \\ &= 1/4 + (-1/4)1/2 = 1/8 \end{aligned}$$