SDS321 Practice Problems for Exam 1 : Solutions

1. A bag contains 8 pairs of shoes; each pair is a different style. You pick two random shoes from the bag.

   (a) What is the probability that the two shoes you picked out are a pair; i.e. left and right of the same style?
   There are C(16,2) ways of choosing two shoes out of 16.
   Out of all possible pairs that can be chosen, 8 pairs are true pairs (i.e., of the same style)
   So the probability is 8/C(16,2) = 1/15
   (or)
   If there are no restrictions:
   There are 16 ways of picking the first shoe, and once it is picked, there are 15 ways of picking the second shoe. So 16*15
   To pick out a pair:
   There are 16 ways of picking the first shoe. Once it's picked, there is only one way of picking its pair. So 16*1
   So answer is 16*1/(16*15)=1/15

   (b) What is the probability you picked one left shoe and one right shoe?
   Denominator: Same as in (a).
   Numerator: There are 16 ways of picking the first shoe. Once it's picked, there are only 8 ways of picking the second shoe since it cannot be similar to the first.
   So answer is 16*8/(16*15)=8/15
   (or)
   Number of ways of picking one left shoe = C(8,1) = 8
   Number of ways of picking one left shoe = C(8,1) = 8
   Number of ways of picking one left and one right = 8*8
   Denominator = C(16, 2) = 16*15/2 = 8*15
   So probability of one right and one left shoe = (8*8)/(8*15) = 8/15

2. How many solutions are there to the inequality $x_1 + x_2 + x_3 \leq 11$, where x_1, x_2, x_3 are non-negative integers.
   Note that in class we solved a version of this problem with an equality. i.e., we solved x1 + x2 + x3 = 11.
   To bring it into that format, let's add an auxiliary variable x4, so that the equation becomes x1 + x2 + x3 + x4 =11. Constraint is $x_i \geq 0$.
   So solution is C(11+4-1, 4-1)
   = C(14, 3) or C(14,11).

3. Suppose a department has 10 men and 15 women. How many ways can we form a committee with six members, if it must have more women than men?
   If w=women, and m=men, committee can have:
   (6w, 0m) or (5w, 1m) or (4w, 2m)
   # ways is C(15,6)+ C(15,5)C(10,1)+ C(15,4)C(10,2)

4. How many ways are there to choose a half dozen donuts from 10 varieties

   a) If there are no two donuts of the same variety.
   So you need to select 6 varieties without replacement from 10 varieties: C(10,6)

   b) If there are at least two varieties.
   # ways of choosing at least 2 varieties =
   (# ways of choosing 6 donuts without any restrictions –
   # ways of choosing 6 donuts such that they are all of the same variety).
   # ways of choosing 6 donuts without any restrictions = the number of solutions to the equation $x1 + x2 + \ldots + x10 = 6$, where $x_i \geq 0$. This is C(15, 9).

   # ways of choosing 6 donuts such that they are all of the same variety = C(10,1)
   So answers is C(15,9) - C(10,1)

   c) If there must be at least one but no more than 4 glazed.

   = (Total # ways) – (# ways : no glazed) – (# ways : 5 glazed) – (# ways :6 glazed)
   = C(15, 9) – C(14, 8) – C(9, 1) – 1 = 1992
   [OR]
   = (# ways exactly 1 glazed) + (# exactly 2) + (#exactly 3) + (#exactly 4)
   = C(13,8) + C(12,8) + C(11,8) + C(10,8) = 1992

5. Given a set A = {a,b,c,d,e},
   a) Suppose you create sequences by drawing from A with replacement. How many different sequences of type A of length n>0 exist that contain at most one a?
   at most 1 => zero or 1
   4^n sequences with no "a" in them.
   $n4^{n-1}$ sequences with one "a" in them.
   So answer is $4^n + n4^{n-1}$

   b) How many subsets of A are there? 2^5
   c) How many non-empty subsets of A are there? 2^5-1
   d) How many subsets of size 3 can you create from A? C(5,3)
   e) How many subsets of A are there that are entirely vowels or entirely consonants? (Assume empty can be considered as entirely vowels or as entirely consonants). 2^2+2^3-1.
   You subtract 1 because the empty set shows up in both lists, so has to be removed once.
   f) How many subsets of A are there that have at least one vowel and one consonant?
   $2^5 - (2^2+2^3-1)$
   g) How many subsets of A of size 3 contain exactly one vowel. C(2,1)C(3,2)
   h) How many ways can you arrange the letters in A? 5!
   i) How many ways can the letters of A be arranged so that all of the vowels are together?
   Glue the two vowels together. Now you have 4 objects to arrange. This can be done in 4! ways. Within the two vowels, they can be arranged in 2! ways. So answer = 2!4!

   j) How many ways can you arrange the letters of A so that it is not the case that all of the vowels are together? 5!- 2!4!

k) How many ways can you arrange the letters in A so that vowels and consonants alternate and the arrangement begins with a consonant. 3!2!

l) For n>0, assume all strings of length *n* from the set *A* (allowing repetition) are equally likely.

i. What is the probability that such a string has no *a*? $4^n/5^n$

ii. What is the probability that such a string has no *b* given that it has no *a*? $3^n/4^n$

6. How many distinct permutations are there of the letters in "perfect"? 7!/2!

7. Billy takes two tests in his probability class. The probability that Billy would pass at least one test is 0.9. The probability that he passes both tests is 0.7. The tests are of equal difficulty (that is, the probability that Billy passes test 1 is the same as the probability that he passes test 2.) What is the conditional probability of Billy passing test 2 given the event that he passes test 1? 7/8 (use Venn diagram to solve)

8. Three persons roll a fair 4-sided die once. Let B_{ij} be the event that person i and person j roll the same face. Show that the events B_{12}, B_{13}, and B_{23} are pairwise independent but are not independent.

   $P(B_{12})= P(B_{13})= P(B_{23})=4*(1/4)*(1/4)=1/4$
   $P(B_{12}\cap B_{13})=P(\text{all three rolled same})=4*(1/4)*(1/4)*(1/4)=1/16= P(B_{12})P(B_{13})$
   $P(B_{12}\cap B_{23})=P(\text{all three rolled same})= 4*(1/4)*(1/4)*(1/4)=1/16= P(B_{12})P(B_{23})$
   $P(B_{13}\cap B_{23})=P(\text{all three rolled same})= 4*(1/4)*(1/4)*(1/4)=1/16= P(B_{13})P(B_{23})$
   But
   $P(B_{12}\cap B_{13}\cap B_{23})=P(\text{all three rolled same})= 4*(1/4)*(1/4)*(1/4)=1/16 \neq$
   $P(B_{12})P(B_{13})P(B_{23})$ so the events are pairwise independent but not independent.

9. Let $P(A)=.5$, $P(B)=.6$, and $P(A\cap B^c)=.2$. Are A and B independent? What is $P(A\cap B|A\cup B)$?

   $P(A\cap B^c)=.2 = P(A)P(B^c) = 0.5*0.4$
   So A and B^c are independent. Therefore A and B are also independent.
   What is $P(A\cap B|A\cup B)$? 3/8 (draw a Venn diagram to help figure it out)

10. If $P(A)=0.4$, $P(B)=0.3$ and $P((A\cup B)^c)=0.42$. Are A and B independent?

    $P(A\cup B) = 1-P(A\cup B^c) = 0.58$, $P(A\cap B) = P(A)+P(B)-P(A\cup B) = 0.12 = P(A) \cdot P(B)$.
    So yes, A and B are independent.

11. Two fair dice are rolled. Define events: A=sum of two dice equals 3, B=sum of two dice equals 7, C=at least one of the dice shows a 1.
    What is P(A|C)? What is P(B|C)? Are A and C independent?

Solution: Each outcome is equally likely, with probability 1/36.

$$A = \{(1,2),(2,1)\}$$
$$B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$
$$C = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)\}.$$
$$P(A|C) = \frac{P(A\cap C)}{P(C)} = 2/11 \neq P(A), \text{ not independent.}$$
$$P(B|C) = \frac{P(B\cap C)}{P(C)} = 2/11.$$

12. Four graduating seniors Alice, Bob, Charles and David are scheduled for job interviews at Random Sampling, Inc at the same time. They've been assigned by the personnel manager to four separate rooms. But they are not aware of this, and randomly assign themselves to the four rooms. What's the probability that none of them ends up in the correct room?

    Total possible permutations = 4! = 24
    # ways in which exactly one person is in the correct room:

    C(4,1) ways in which a person can be in the correct room. After this person is fixed, there are two ways to incorrectly assign people to the remaining 3 rooms. After these two people are fixed, there is only one way to incorrectly assign the remaining 2 people to the 2 rooms. So this is C(4,1)*2 = 8

    # ways in which two people are in the correct room = C(4,2) = 6. Use same logic as above.
    # ways three people are in the correct room = 0
    # ways all are in the correct room = 1
    So # ways in which no one is in the correct room = 24 – (8+6+0+1) = 9
    Probability of this = 9/24

13. Three molecules of type A, three of type B, and three of type C are to be linked together to form a chain molecule. One example of such chain molecule is ABCABCABC.

    (a) How many such chain molecules are there? (molecules of same type are indistinguishable.)

    9! / 3!3!3! = 1680

    (b) If the link between consecutive molecule As is unstable, how many stable chain molecules can be formed?

    = Total # possible – [(# ways 2 As are together) - (# ways in which 3 As are together)]

    = (9! / 3!3!3!) – (8! / 3!3!) + (7! / 3!3!) = 700

    This is because, when we list the # ways 2 As can appear together, it counts those cases where the 3 As appear together twice. So we need subtract it once.

    Let's look at this with a simpler example, with just 3 As and 3 Bs.

stable arrangements would be the # arrangements where As don't appear together, and from the list below, we can see that this is 4.

Example with just 3 A's & 3 B's
we have 20 total arrangements :

3 A's together = 4	just 2 A's together = 12			No A's together = 4
AAABBB	AABABB	ABBAAB	BBAABA	ABBABA
BAAABB	AABBAB	ABBBAA	BABBAA	ABABAB
BBAAAB	AABBBA	BBABAA	BAABAB	BABABA
BBBAAA	ABAABB	~~BBBAAA~~	BAABBA	ABABBA
			BABAAB	

So there are 4 arrangements where A's don't appear together.

$\frac{6!}{3!3!} = 20 \rightarrow$ total # arrangements

$\frac{5!}{3!} = 20 \rightarrow$ # arrangements where 2 A's appear together. This is not 12 but 20 because the 4 ways in which 3 A's appear together are being counted twice.

So $12 + (2 \times 4) = 20$

Example : $\boxed{AA}$ A BBB, A $\boxed{AA}$ BBB > counted twice

$\frac{4!}{3!} = 4 \rightarrow$ # ways 3 A's appear together

So answer is $\frac{6!}{3!3!} - \left[\frac{5!}{3!} - \frac{4!}{3!}\right]$

14. On Alex's commuting route there are three intersections with traffic signals. The probability that she must stop at the first signal is 0.4. If she stops at one signal, then the next signal is always green when she gets there. If she doesn't stop at one signal, the probability of her stopping at the next signal is 0.5.

(a) What is the probability she arrives school without stopping at signals.
= P(no stop at 1^{st})P(no stop at 2^{nd}|no stop at 1^{st})P(no stop at 3^{rd}|no stop at 2^{nd})
= 0.6 * 0.5 * 0.5 = 0.15

(b) What is the probability that she stops at exactly one signal given the second signal is

green?
= P(stops at exactly one | no stop at 2^{nd})
= P(stops at exactly one AND no stop at 2^{nd}) / P(no stop at 2^{nd})
Numerator events: (stop, no stop, no stop) OR (no stop, no stop, stop)
Numerator probability = (0.4*1*0.5) + (0.6*0.5*0.5) = 0.2 + 0.15 = 0.35

Denominator events and probabilities:
P(stop, no stop, stop) = 0.4*1*0.5 = 0.2
P(stop, no stop, no stop) = 0.2
P(no stop, no stop, stop) = 0.15
P(no stop, no stop, no stop) = 0.15
Total denominator probability = 0.7

So answer = 0.35/0.7 = 0.5

15. Alex is new to basketball. He is learning to shoot the ball through the hoop from the free throw line, which he is currently able to do 20% of the time. He goes to the gym with the intention of practicing just his free throws.

a) Suppose he is unable to get the ball through the hoop in his first 12 attempts. What is the probability that he will be able to make it for the first time, on his 15th attempt?

X = number of attempts until first successful shot ~ Geo(0.2)
$P(X=15 \mid X>12) = P(X=12+3 \mid X>12) = P(X=3) = 0.8 \times 0.8 \times 0.2 = 0.128$

b) Suppose he is unable to get the ball through the hoop in his first 12 attempts. What is the probability that he will need at least 15 attempts to make his first successful shot?

$P(X \geq 15 \mid X > 12) = P(X > 14 \mid X > 12) = P(X > (12+2) \mid X > 12) = P(X > 2)$
$= 1 - P(X=1) - P(X=2)$
$= 1 - 0.2 - (0.8 \times 0.2)$
$= 1 - 0.2 - 0.16 = 0.64$

16. Suppose 20% of people in a city are estimated to have a disease. You test 20 individuals chosen at random to see if they have the disease. What is the probability that at least two of the people test positive for the disease?

X = # test +ive ~ Bin(20, 0.2)
$P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - (0.8)^{20} - [20 \times (0.2) \times (0.8)^{19}]$

17. The number of typos in textbooks printed by publishing company XYZ follows a Poisson distribution. It is known that the mean number of typos in 100 pages of their textbooks is 3.2. What is the probability of exactly 10 typos in a 200-page textbook?

Mean number of typos in 200 pages = 3.2*2 = 6.4

Ans = $(6.4)^{10}$*exp(-6.4)/10! = 0.053