SDS 321: Practice questions

1. How many unique combinations can you get by rearranging the letters MISSISSIPPI?

   11 letters so 11! permutations. 4 Is, 4Ss, 2 Ps, so 11!/(4!4!2!) unique combinations.

2. On the first day of a non-leap-year, I put \$1 in a box. On the second day, I put \$2 in the box. On the third day, I put \$3 in. And so on. At the end of the year (365 days), how much money is in the box?

   The first day you put in \$1, the last day you put in \$365. The average of these two is 183. The average of the second and penultimate days is also 183. Etc. So, the total is $365 \times 183 = \$66795$.

3. Let X be a normal random variable with mean 3 and variance 1, and let Y be a normal random variable with mean 4 and variance 2.

   (a) What is the distribution of $Z = X + Y$? $Normal(7, 3)$

   (b) What is the probability that Z is between 6 and 8? $\mathbf{P}(6 \leq Z \leq 8) = \mathbf{P}(\frac{6-7}{\sqrt{3}} \leq \frac{Z-7}{\sqrt{3}} \leq \frac{8-7}{\sqrt{3}}) = \mathbf{P}(-0.577 \leq \frac{Z-7}{\sqrt{3}} \leq 0.577) = 0.44$ (from standard normal tables)

4. I am waiting for a bus, that I know will arrive at some time between 1pm and 2pm, with all times being equally likely. It gets to 1:30, and the bus has still not arrived. What is the probability that it arrives before 1:40? 1/3

5. Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 0.125x + 0.125 & -1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

   What is the PDF of $Z = |X|$? The CDF is $P(Z \leq z)$. If $z < 0$ this is zero since Z is always positive. If $z > 3$ this is one. So the action is in between.

   If $0 \leq z \leq 1$, then

$$F_Z(z) = P(|X| \leq z) = P(-z \leq X \leq z) = \int_{-z}^{z} (0.125x + 0.125)dx = .25z.$$

   If $1 < z \leq 3$ then

$$F_Z(z) = P(|X| \leq z) = P(-1 < X < 1) + P(1 \leq X \leq z) = .25 + .125(z^2/2 - 1) + .125(z - 1)$$

Differentiating we get:

$$f_Z(z) = \begin{cases} 0.25 & 0 \leq z \leq 1 \\ 0.125 + 0.125z & 1 < z \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

6. Alice and Bob are playing rock-paper-scissors. If both Alice and Bob play the same hand, they play again. What is the expected number of turns before someone wins?

   This is a Geometric distribution with $p = 2/3$ (probability someone wins). The expected value of a random variable with total number of tries upto the first success is $1/p = 3/2$. Since we are interested in the expected number of turns *before* somebody wins, our answer is $3/2 - 1 = 1/2$.

7. On a given day, a $Poisson(100)$ number of insects fly through my yard. Using an appropriate approximation, what is the the probability that, over the month of May (31 days), the average number of insects is between 98 and 102? You may use the fact that a $Poisson(\lambda)$ random variable has mean and variance λ.

   $E[X_i] = \lambda$ so $E[\bar{X}] = \lambda$. $\text{var}(X_i) = \lambda$, so $\text{var}(\bar{X}) = \lambda/n$. We can approximate the distribution as $Normal(\lambda, \lambda/n) = Normal(100, 100/31)$.

$$\mathbf{P}(98 \leq \bar{X} \leq 102) \approx \mathbf{P}(\frac{-2}{\sqrt{100/31}} \leq Z \leq \frac{2}{\sqrt{100/31}} = 0.73$$

8. Combinatorics question:

   (a) How many different solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where x_1, x_2 and x_3 are positive integers? (count "$x_1 = 1, x_2 = 2, x_3 = 7$" and "$x_1 = 2, x_2 = 1, x_3 = 7$" as two separate solutions). Stars and bars... We have 9 places to put the first bar, and 8 places to put the second. But, there are 2 possible rearrangements of the bars. So, $9 \times 8/2 = 36$. Or, equivalently, there are $\binom{9}{2=36}$ ways of placing 2 bars.

   (b) How many different solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where $x_1 < x_2$? We know there are 36 solutions in total. Of these, let's remove the solutions where $x_1 = x_2$. We have 4 such solutions ($x_1 = x_2 = 1, x_1 = x_2 = 2, x_1 = x_2 = 3, x_1 = x_2 = 4$), leaving 32 solutions with different x_1 and x_2. Of these, half have $x_1 < x_2$, so 16 solutions.

   (c) How many different solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where $x_1 < x_2 < x_3$? We know there are 36 solutions in total. Let's first remove the solutions with repeats. We have no solutions with all three numbers the same, and 12 solutions with 2 numbers the same (3 ways each of having 2 ones, 2 twos, 2 threes, 2 fours). So, 24 solutions with no repeats. Of these, there are 3! permutations of each sequence, so there are $24/3! = 4$ solutions. Double checking, we have 1+2+7, 1+3+6, 1+4+5, 2+3+5... that's it!

9. Let X be a continuous random variable, with PDF:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 0.5 & 0 \le x < 1 \\ ce^{-x} & x \ge 1 \end{cases}$$

(a) What is c? We need $\int_0^\infty f_X(x)dx = 1$, so $\int_1^\infty ce^{-x}dx = 0.5$. The integral gives $c[-e^{-x}]_1^\infty = ce^{-1}$ so $c = 0.5e^1$

(b) What is the conditional expectation of X, given $X < 1$?

$$E[X|X < 1] = 0.5$$

(c) What is the conditional expectation of X, given $X \ge 1$? $E[X1(X \ge 1)] = \int_1^\infty cxe^{-x}dx = 1$, since ce^{-x} is $f_X(x)1(x \ge 1)$.
Note that

$$E[X|X \ge 1] = E[X1(X \ge 1)]/P(X \ge 1) = 1/(1/2) = 2$$

Another way to do this is to note that

$$f_{X|X\ge 1}(x) = f_X(x)1(X \ge 1)/P(X \ge 1) = ce^{-x}/0.5$$

and so

$$E[X|X \ge 1] = \int_1^\infty cxe^{-x}dx/0.5 = 2c\int_1^\infty xe^{-x}dx$$

Substituting, x-1=v

$$2c\int_0^\infty (1+v)e^{-(1+v)}dv = \frac{2c}{e}(1 + \int_0^\infty ve^{-v}dv) = 4c/e = 2$$

(d) What is the expectation of X?

$$E[X] = P(X < 1) \times 0.5 + P(X \ge 1) \times 2 = 0.5 \times 0.5 + 0.5 \times 2 = 1.25$$

10. Let X and Y be random variables with joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} \frac{ay}{x^2} & x \ge 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is a?

$$f_X(x) = \int_0^1 \frac{ay}{x^2}dy = \left[\frac{ay^2}{2x^2}\right]_0^1 = \frac{a}{2x^2}$$

$$1 = \int_1^\infty \frac{a}{2x^2}dx = \left[-\frac{a}{2x}\right]_1^\infty = 1/2$$

so, $a = 2$

(b) What is the conditional PDF $f_{Y|X}(y|x)$ of Y given $X = x$?

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{ay/x^2}{a/2x^2} = 2y$$

(c) What is the conditional expectation of Y given X?

$$E[Y|X = x] = \int_0^1 2y^2 dy = 2/3$$

So, $E[Y|X] = 2/3$.

(d) What is the expected value of Y? $E[Y] = E[E[Y|X]] = 2/3$

11. I have three envelopes, each containing two objects. In one, there is a silver square and a gold disk. In another, there are a gold square and a gold disk. In the third, there are a silver square and a silver disk.

(a) I pick an envelope (at random), and take out an object (at random). It is gold. What is the probability that the second object is silver?

We have 4 possibilities: (S,S) (prob 1/3), (S,G) (prob 1/6) , (G,S) prob (1/6) , (G,G) (prob 1/3). We want $A = \{\text{2nd item is S}\}$ and $B = \{\text{1st item is G}\}$. $P(A|B) = P(AB)/P(B) = \frac{P(GS)}{P(G,S)+P(G,G)} = (1/6)/(1/2) = 1/3$.

(b) I put the objects back in their envelope and shuffle the envelopes. I again pick an envelope (at random), and take out an object (at random). It is a gold disk. What is the probability that the second object is silver?

Let D be disk, Q be square. Our sample space, after we know the first object is a gold disk, is now $(GD, SQ), (GD, GQ)$. So, the probability is 1/2.

12. Let X and Y be uniform random variables between 0 and 1. What is the probability that:

(a) $X < Y$ 0.5

(b) $X < 2Y$ If $Y = y$, $\mathbf{P}(X \leq 2y) = 2y$ if $y \leq 0.5$, 1 if $y > 0.5$, . So, overall, $\mathbf{P}(X \leq 2Y) = \int_0^{0.5} 2y dy + \int_{0.5}^1 dy = 0.25 + 0.5 = 0.75$.

(c) $X + Y < 0.5$ If $Y = y$, $\mathbf{P}(X + Y < 0.5|Y = y) = 0.5 - y$ for $y < 0.5$. Integrating over y we have:

$$\mathbf{P}(X + Y < 0.5) = \int_0^{0.5} (0.5 - y)dy = 0.125$$

(d) $\max\{X, Y\} \leq 0.7$ $\mathbf{P}(\max\{X, Y\} \leq 0.7 = \mathbf{P}(X \leq 0.7)\mathbf{P}(Y \leq 0.7) = 0.7^2 = 0.49$

13. If there are no distractions, it takes me 30 minutes to walk to the store. However, if I pass someone with a dog, I stop and pet the dog and chat to the owner. The number Y of dogs I pass is a Poisson random variable with mean 2. Each time I stop, the number of minutes I spend petting the dog and chatting is an exponential random variable with PDF:

$$f_X(x) = 0.5e^{-0.5x}$$

(a) If I see a single dog, what is the expectation and variance of the time spent petting the dog and chatting to its owner? $E[X] = \int_0^\infty 0.5xe^{-0.5x}dx = 2$, $E[X^2] = \int_0^\infty 0.5x^2e^{-0.5x} = 8$, $\text{var}(X) = E[X^2] - E[X]^2 = 4$.

(b) What is the conditional expectation of the total time spent petting dogs and chatting to their owners, as a function of Y? $E[X|Y = y] = 2y$, (expectation of sum), so $E[X|Y] = 2Y$

(c) What is the conditional variance of the total time spent petting dogs and chatting to their owners, as a function of Y? $\text{var}(X|Y = y) = 4y$ (variance of sum of independent random variables), so $\text{var}(X|Y) = 4Y$

(d) What is the expectation and variance of the total time it takes me to get to the store? You may use the fact that the variance of a Poisson random variable is the same as its mean. $E[X] = E[E[X|Y]] = E[2Y] = 2E[Y] = 4$. $\text{var}(X) = E[\text{var}(X|Y)]+\text{var}(E[X|Y]) = E[4Y]+\text{var}(2Y) = 4E[Y]+4\text{var}(Y) = 4\times2+4\times2 = 16$.

14. My partner and I are one of 10 married couples at a dinner party. The 20 people are given random seats around a round table.

(a) What is the probability that I am seated next to my spouse? there are two seats next to me, and 19 people who can sit in those seats. $P(\text{spouse on left}) = 1/19$, $P(\text{spouse on right}|\text{spouse not on left})P(\text{spouse not on left}) = 1/18 \times 18/19 = 1/19$, so $P(\text{next to spouse}) = 2/19$.

(b) What is the expected number of couples that are seated next to each other? The expected number of people sat next to their spouse is $40/19$ (sum of expectations). So, the expected number of couples sat next to each other is $20/19$.

Standard normal table

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990